Notes on Integrable Probability

Galobel Wang
The University of Hong Kong

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Abstract

This article consists of some notes taken by the author while studying integrable probability. References: [1], [2], [3], [4], [5], [6], [7], [8]

This part, let me introduce some concepts in representation theory, and our goal is **Cauchy Identity** of Schur polynomials:

Theorem 0.1 (Cauchy Identity). *We have*

$$\sum_{\lambda \in \mathcal{Y}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

and also

$$\sum_{\lambda \in \mathcal{Y}} \frac{p_\lambda(x_1, x_2, \ldots) p_\lambda(y_1, y_2, \ldots)}{z_\lambda} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(x_1, x_2, \ldots) p_k(y_1, y_2, \ldots)}{k}\right) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

where $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$ *and* $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$, *where* $m_i(\lambda)$ *is the number of rows of length* i *in* λ *and* λ *is a partition.*

Cauchy Identity describes the decomposition of symmetric algebra over the tensor product representation of $\mathbf{GL}(n) \times \mathbf{GL}(n)$.

Contents

1 Topics in Representation Theory

1.1 Partitions and Young diagrams

Definition 1.1 (partition)**.** *A partition is a sequence of non-negative integers* $\lambda = (\lambda_1, \lambda_2, \ldots)$ *such that* $\lambda_1 \geq \lambda_2 \geq \cdots$ *and containing only finitely many non-zero elements.*

We regard $(3, 1)$, $(3, 1, 0)$, $(3, 1, 0, 0, \ldots)$ as the same partition.

Different quantities related with partitions needs a definition.

Definition 1.2. *Let* $\lambda = (\lambda_1, \lambda_2, \ldots)$ *be a partition.*

- *The non-zero entries* λ_i *of* λ *are called the parts of* λ*.*
- *The number of parts is called the length of* λ*. We denote it by* $\ell(\lambda)$*.*
- *The sum of the parts* $|\lambda| := \sum_i \lambda_i$ *is called the weight of* λ*.*
- *If* $|\lambda| = n$ *we say that* λ *is a partition of* n*.*

We denote by $\mathcal{P}_n$ the set of all partitions of n, by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}_0$ the set containing only the empty partition $\varnothing$. A concise notation for a partition is given by $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ with $m_i = \mathrm{Card}\{j : \lambda_j = i\}$.

There is a nice graphical representation of the partition in terms of Young diagrams.

Definition 1.3 (Young diagram)**.** *A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order representing the parts of a partition. An example is represented in Figure 1. We denote by* $\mathcal{Y}$ *the set of all Young diagrams.*

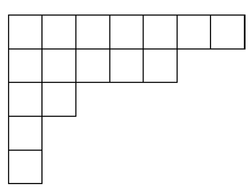


Figure 1: The Young diagram associated to the partition $\lambda = (7, 5, 2, 1, 1)$

Definition 1.4 (Young tableau)**.** *A Young tableau is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet, which is usually required to be a totally ordered set.*

- *A tableau is called* ***standard*** *if the entries in each row and each column are increasing.*

- *A tableau is called* ***semistandard****, or column strict, if the entries weakly increase along each row and strictly increase down each column.*

Listing the number of boxes of a Young diagram in each column gives another partition, the conjugate partition of λ; one obtains a Young diagram of that shape by reflecting the original diagram along its main diagonal (see Figure 2). In formulas,

$$\lambda'_i = |\{j \mid \lambda_j \geq i\}|$$

from which one has the identity $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$.

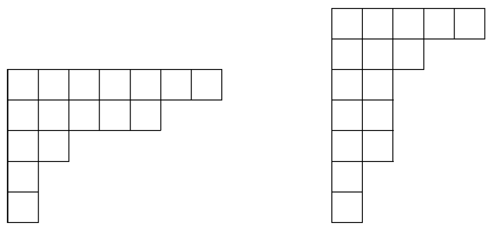


Figure 2: The conjugate λ' of the Young diagram in Figure 1; λ' is a Young diagram associated to the partition $(5, 3, 2, 2, 2, 1, 1)$.

Definition 1.5. *We define now a partial ordering on partitions: given two partitions λ, μ, we say that $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for any $i \geq 1$. If $\lambda \subset \mu$ we can define the skew diagram, the set-theoretic difference $\theta = \mu - \lambda$. We set $|\theta| := |\mu| - |\lambda| = \sum_i (\mu_i - \lambda_i)$.*

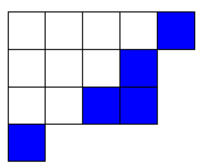


Figure 3: Skew diagram (blue area) $\theta = (1, 1, 2, 1)$ for the partition $\mu = (5, 4, 4, 1)$ and $\lambda = (4, 3, 2)$.

An example of skew diagram is represented in Figure 3.

Definition 1.6 (skew diagram)**.** *A skew diagram θ is a horizontal m-strip (resp. vertical m-strip) if the total number of extra boxes is exactly m and $\theta'_i \leq 1$ (resp. $\theta_i \leq 1$) for any i. A horizontal (resp. vertical) strip has at most one square in each column (resp. row).*

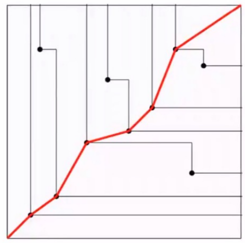
Remark 1.1. *If $\theta = \mu - \lambda$, then θ is a horizontal (resp. vertical) strip iff*

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \cdots$$

In this case we say that λ and μ interlace and we use the notation $\lambda \lhd \mu$ or $\mu \rhd \lambda$.

1.2 Example: Robinson-Schensted correspondence

Longest Increasing subsequences in a random permutation

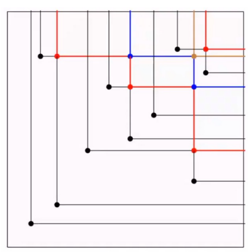


- The Hammersley Process can be translated to: "The number of non-intersecting broken lines that can be selected in the chosen region.
- Order the horizontal and vertical coordinates of the Poisson points and write the coordinates (x, y) of each point in the form of a biletter $\binom{x}{y}$
- The coordinates here depends only on the order of appearance and is independent of the Cartesian coordinate system and distances.
- Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 9 & 2 & 4 & 7 & 5 & 6 & 10 & 3 & 8 \end{pmatrix} \tag{1.1}$$

- The Hammersley Process (red line) corresponds to the longest increasing subsequence in the sequence of y coordinates.

Robinson-Schensted correspondence: geometric description



The purpose is to find a pair of Young diagrams that correspond one-to-one with the permutation (1.1).

The red points are the intersections where the black points emit rays upward and to the right, while the blue points are the focal points of the rays emitted by the red points, and so on.

1.3 Symmetric functions

1.3.1 Basic notion

In this section we briefly present the algebra Λ of symmetric functions in infinitely many variables. Let $\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$ be the space of polynomials in $x_1, \ldots, x_N$ which are symmetric under the action of the semigroup S_N, which means with respect to permutations of the x_i's. Λ_N is a graded ring, in fact

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k,$$

where Λ_N^k consists of homogeneous symmetric polynomials of degree k, together with the zero polynomial. Constant are in the 0th graded component.

Consider the homomorphism $\pi_{N+1} : \mathbb{C}[x_1, \ldots, x_{N+1}] \to \mathbb{C}[x_1, \ldots, x_N]$, which sends x_{N+1} to zero and the other x_i's to themselves. This maps elements of Λ_{N+1} to elements of Λ_N and it generates a so-called tower of graded algebras:

$$\mathbb{C} \xleftarrow{\pi^1} \Lambda_1 \xleftarrow{\pi^2} \Lambda_2 \xleftarrow{\pi^3} \cdots$$

Finally one sets

$$\Lambda := \varprojlim_{N \to \infty} \Lambda_N = \{(f_1, f_2, \ldots) | f_i \in \Lambda_i, \pi_i f_i = f_{i-1}, \deg(f_i) < \infty\}$$

Example 1.1.

- *$p_1 = x_1 + x_2 + \ldots$ belongs to Λ,*
- *$p_2 = (1 + x_1)(1 + x_2) \ldots$ does not belong to Λ, since it has infinite degree,*
- *$p_3 = x_1x_2 + x_1x_3 + \ldots + x_2x_3 + \ldots$ belongs to Λ,*
- *$p_4 = 1$ belongs to Λ,*
- *$p_5 = x_1 + x_2^2 + x_3 + x_4^2 + \ldots$ does not belong to Λ, since it is not symmetric.*

Remark 1.2. *One can think also of elements of Λ as formal power series in infinitely many variables $x_1, x_2, \ldots$ of bounded degree which are invariant under permutations of the x_i's.*

In order to work with symmetric functions, it is useful to know some basis of Λ. Let us define three useful symmetric functions.

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Let λ be any partition of length $\leq n$. The polynomial

$$m_\lambda(x_1, \dots, x_n) = \sum_{\text{over all distinct permutations } \alpha \text{ of } \lambda=(\lambda_1,\dots,\lambda_n)} x^\alpha$$

is clearly symmetric, and the m_λ (as λ runs through all partitions of length $\leq n$) form a $\mathbb{Z}$-basis of Λ_n. Hence the m_λ such that $\ell(\lambda) \leq n$ and $|\lambda| = k$ form a $\mathbb{Z}$-basis of Λ_k^n; in particular, as soon as $n > k$, the m_λ such that $|\lambda| = k$ form a $\mathbb{Z}$-basis of Λ_k^n.

Definition 1.7. *1. For $n \geq 1$, the nth elementary symmetric function e_n is defined by*

$$e_n = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} \cdots x_{i_n}.$$

We will also use the convention $e_0 = 1$.

2. For $n \geq 1$, the nth complete homogeneous function h_n is defined by

$$h_n = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

We will also use the convention $h_0 = 1$.

3. For $n \geq 1$, the nth power sum p_n is defined by

$$p_n = \sum_{i \geq 1} x_i^n.$$

It can be proven that each of these set are basis of Λ.

Theorem 1.1. *The systems $\{e_n\}, \{h_n\}, \{p_n\}$ are algebraically independent generators of Λ, thus $\Lambda = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$.*

Elementary symmetric functions

For each integer $r \geq 0$ the rth elementary symmetric function e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = m_{(1^r)}$$

for $r > 1$. The generating function for the e_r is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i > 1} (1 + x_i t). \tag{1.2}$$

(t being another variable), as one sees by multiplying out the product on the right. (If the number of variables is finite, say n, then e_r (i.e. $\rho_n(e_r)$) is zero for all $r > n$, and

(1.2) then takes the form

$$\sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n}(1 + x_i t),$$

both sides now being elements of $\Lambda_n[t]$. Similar remarks will apply to many subsequent formulas, and we shall usually leave it to the reader to make the necessary (and obvious) adjustments.

Theorem 1.2. *For each partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ *define*

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$

Let λ' *be a partition,* λ' *its conjugate. Then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_\mu m_\mu, \tag{1.3}$$

where the a_μ *are non-negative integers, and the sum is over partitions* $\mu < \lambda$ *in the natural ordering.*

Proof 1.1. *When we multiply out the product* $e_{\lambda'} = e_{\lambda_1} e_{\lambda_2} \cdots$*, we shall obtain a sum of monomials, each of which is of the form*

$$(x_{i_1} x_{i_2} \cdots)(x_{j_1} x_{j_2} \cdots) \ldots = x^\alpha,$$

say, where $i_1 < i_2 < \cdots < i_{r'}, j_1 < j_2 < \cdots < j_{r''}$*, and so on. If we now enter the numbers* $i_1, i_2, \ldots, i_k$ *in order down the first column of the diagram of* λ*, then the numbers* $j_1, j_2, \ldots, j_{k'}$ *in order down the second column, and so on, it is clear that for each* $r \geq 1$ *all the symbols* $< r$ *so entered in the diagram of* λ *must occur in the top* r *rows. Hence* $\alpha_1 + \cdots + \alpha_r \leq \lambda_1 + \cdots + \lambda_r$*, for each* $r \geq 1$*, i.e. we have* $\alpha \leq \lambda$*. It follows that*

$$e_{\lambda'} = \sum_{\mu < \lambda} a_\mu m_\mu$$

with $a_\mu > 0$ *for each* $\mu \leq \lambda$*, and the argument above also shows that the monomial* x^α *occurs exactly once, so that* $a_\lambda = 1$*.*

Theorem 1.3. *We have*

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$$

and the e_r *are algebraically independent over* $\mathbb{Z}$*.*

Proof 1.2. *The* m_λ *form a* $\mathbb{Z}$*-basis of* Λ*, and (1.3) shows that the* e_λ *form another* $\mathbb{Z}$*-basis: in other words, every element of* Λ *is uniquely expressible as a polynomial in the* e_r*.*

Remark 1.3. *When there are only finitely many variables $x_1, \ldots, x_n$, 1.3 states that $\Lambda_n = \mathbb{Z}[e_1, \ldots, e_n]$, and that $e_1, \ldots, e_n$ are algebraically independent. This is the usual statement of the 'fundamental theorem on symmetric functions'.*

Complete symmetric functions

For each $r > 0$ the rth complete symmetric function h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$, so that

$$h_r = \sum_{|\lambda|=r} m_\lambda.$$

In particular, $h_0 = 1$ and $h_1 = e_1$. It is convenient to define h_r and e_r to be zero for $r < 0$.

The generating function for the h_r is

$$H(t) = \sum_{r\geq 0} h_r t^r = \prod_{i>1} (1 - x_i t)^{-1}. \tag{1.4}$$

To see this, observe that

$$(1 - x_i t)^{-1} = \sum_{k\geq 0} x_i^k t^k,$$

and multiply these geometric series together.

From (1.2) and (1.4) we have

$$H(t)E(-t) = 1 \tag{1.5}$$

or, equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \tag{1.6}$$

for all $n \geq 1$.

Since the e_r are algebraically independent 1.3, we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

by

$$\omega(e_r) = h_r.$$

for all $r > 0$. The symmetry of the relations (1.6) as between the $esandthe$hs shows that

$$\omega \text{ is an involution, i.e. } \omega^2 \text{ is the identity map.} \tag{1.7}$$

It follows that ω is an automorphism of Λ, and hence from 1.3 that

$$\Lambda = \mathbb{Z}[h_1, h_2, \ldots] \tag{1.8}$$

and the h_r are algebraically independent over $\mathbb{Z}$.

Remark 1.4. *If the number of variables is finite, say n (so that $e_r = 0$ for $r > n$) the mapping $\omega : \Lambda_n \to \Lambda_n$ is defined by $\omega(e_r) = h_r$ for $1 \le r \le n$, and is still an involution by reason of (1.6); we have $\Lambda_n = \mathbb{Z}[h_1, \ldots, h_n]$ with $h_1, \ldots, h_n$ algebraically independent, but $h_{n+1}, h_{n+2}, \ldots$ are non-zero polynomials in $h_1, \ldots, h_n$ (or in $e_1, \ldots, e_n$).*

As in the case of the e's, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. The h_λ form a $\mathbb{Z}$-basis of Λ. We now have three $\mathbb{Z}$-bases, all indexed by partitions: the m_λ, the e_λ, and the h_λ, the last two of which correspond under the involution ω. If we define

$$f_\lambda = \omega(m_\lambda)$$

for each partition λ, the f_λ form a fourth $\mathbb{Z}$-basis of Λ. (The f_λ are the 'forgotten' symmetric functions: they have no particularly simple direct description.)

The relations (1.6) lead to a determinant identity which we shall make use of later. Let N be a positive integer and consider the matrices of $N+1$ rows and columns

$$H = (h_{i-j})_{0 \le i,j < N}, \quad E = \left((-1)^{i-j} e_{i-j}\right)_{0 \le i,j < N}$$

with the convention mentioned earlier that $h_r = e_r = 0$ for $r < 0$. Both H and E are lower triangular, with 1's down the diagonal, so that $\det H = \det E = 1$; moreover the relations (1.6) show that they are inverses of each other. It follows that each minor of H is equal to the complementary cofactor of E', the transpose of E.

Let λ, μ be two partitions of length $\le p$, such that λ' and μ' have length $\le q$, where $p + q = N + 1$. Consider the minor of H with row indices $\mu_i + p - i$ $(1 \le i \le p)$. The complementary cofactor of E' has row indices $p - 1 + j - \lambda_j$ $(1 \le j \le q)$ and column indices $p - 1 + j - \mu_j$ $(1 \le j \le q)$. Hence we have

$$\det\left((h_{\lambda_i - \mu_j - i + j})_{1 \le i,j \le p}\right) = (-1)^{|\lambda| + |\mu|} \det\left((-1)^{\lambda_j - \mu_j - i + j} e_{\lambda_i - \mu_j - i + j}\right)_{1 \le i,j \le q}.$$

The minus signs cancel out, and therefore we have

$$\det\left((h_{\lambda_i - \mu_j - i + j})_{1 \le i,j \le p}\right) = \det\left((e_{\lambda_i - \mu_j - i + j})_{1 \le i,j \le q}\right).$$

In particular, taking $\mu = 0$,

$$\det\left((h_{\lambda_i - i + j})\right) = \det\left((e_{\lambda_i - i + j})\right).$$

Power sums

For each $r > 1$ the rth power sum is

$$p_r = \sum x_i^r = m_{(r)}.$$

The generating function for the p_r is

$$\begin{aligned} P(t) &= \sum_{r>1} p_r t^{r-1} = \sum_{i>1} \sum_{r>1} x_i^r t^{r-1} \\ &= \sum_{i>1} \frac{x_i}{1 - x_i t} \\ &= \sum_{i>1} \frac{d}{dt} \log \frac{1}{1 - x_i t}. \end{aligned} \tag{1.9}$$

so that

$$P(t) = -\frac{d}{dt} \log \prod_{i>1} (1 - x_i t)^{-1} = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}. \tag{1.10}$$

Likewise we have

$$P(-t) = -\frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}. \tag{1.11}$$

From above two equations, we obtain

$$nh_n = \sum_{r \geq 1} p_r h_{n-r}, \tag{1.12}$$

$$ne_n = \sum_{r \geq 1} (-1)^{r-1} p_r e_{n-r}. \tag{1.13}$$

for $n > 1$, and these equations enable us to express the h's and the e's in terms of the p's, and vice versa. The equations (1.13) are due to Isaac Newton, and are known as Newton's formulas. From (1.12) it is clear that $h_n \in Q[p_1, \ldots, p_n]$ and $p_n \in \mathbb{Z}[h_1, \ldots, h_n]$, and hence that

$$Q[h_1, \ldots, h_n] = Q[p_1, \ldots, p_n].$$

Since the h_r are algebraically independent over $\mathbb{Z}$, and hence also over $\mathbb{Q}$, it follows that

$$\Lambda_Q = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = Q[p_1, p_2, \ldots]$$

and the p_r are algebraically independent over $\mathbb{Q}$.

Hence, if we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, then the p_λ form a $\mathbb{Q}$-basis of Λ_Q. But they do not form a $\mathbb{Z}$-basis of Λ: for example, $h_2 = \frac{1}{2}(p_1^2 + p_2)$ does not have integral coefficients when expressed in terms of the p_λ.

Since the involution ω interchanges $E(t)$ and $H(t)$ it follows from (1.10) and (1.11) that

$$\omega(p_n) = (-1)^{n-1} p_n$$

for all $n \geq 1$, and hence that for any partition λ we have

$$\omega(p_\lambda) = \epsilon_\lambda p_\lambda$$

where $\epsilon_\lambda = (-1)^{|\lambda| - l(\lambda)}$.

Generating functions are very often useful, thus let us compute define and compute them for our family of polynomials.

Definition 1.8. *Define the following generating functions,*

$$H(z) = \sum_{k \geq 0} h_k z^k, \quad E(z) = \sum_{k \geq 0} e_k z^k, \quad P(z) = \sum_{k \geq 1} p_k z^{k-1},$$

with $e_0 = h_0 = 1$.

Theorem 1.4. *The following identities hold:*

$$H(z) = \prod_{i \geq 1} \frac{1}{1 - x_i z},$$

$$E(z) = \prod_{i \geq 1} (1 + x_i z),$$

$$P(z) = \frac{d}{dz} \sum_{i \geq 1} \ln\left(\frac{1}{1 - x_i z}\right),$$

$$H(z) = \frac{1}{E(-z)} = \exp\left(\sum_{k \geq 1} \frac{z^k}{k} p_k\right).$$

Proof 1.3. *A series expansion of $\frac{1}{1 - x_i z} = 1 + x_i z + (x_i z)^2 + \ldots$ leads to*

$$(1 + x_1 z + x_1^2 z^2 + \ldots) \cdot \left(1 + x_2 z + x_2^2 z^2 + \ldots\right) \cdots$$

$$= 1 + z \sum_{i \geq 1} x_i + z^2 \left(\sum_{1 \leq j < i} x_i x_j + \sum_{i \geq 1} x_i^2\right) + \ldots$$

$$= 1 + zh_1 + z^2 h_2 + \ldots = H(z).$$

For $E(z)$ we have

$$\prod_{i\geq 1}(1 + x_i z) = (1 + x_1 z)(1 + x_2 z)\cdots$$

$$= 1 + z\sum_{i\geq 1} x_i + z^2 \sum_{1\leq i<j} x_i x_j + \cdots$$

$$= 1 + \sum_{k\geq 1} z^k e_k = E(z).$$

A series expansion of $\ln\left(\frac{1}{1-x_i z}\right) = -\ln(1 - x_i z) = x_i z + \frac{(x_i z)^2}{2} + \frac{(x_i z)^3}{3} + \cdots$ *leads to*

$$\sum_{i\geq 1} \frac{d}{dz} \ln\left(\frac{1}{1 - x_i z}\right) = \sum_{i\geq 1}(x_i + x_i^2 z + x_i^3 z^2 + \cdots)$$

$$= p_1(x_1, x_2, \ldots) + zp_2(x_1, x_2, \ldots) + z^2 p_3(x_1, x_2, \ldots) + \cdots = P(z).$$

Finally, we have

$$P(z) = \sum_{i\geq 1} p_i z^{i-1} = \frac{d}{dz}\sum_{i\geq 1} \ln\left(\frac{1}{1 - x_i z}\right)$$

$$= \frac{d}{dz} \ln\left(\prod_{i\geq 1} \frac{1}{1 - x_i z}\right) = \frac{d}{dz} \ln H(z),$$

which after integrating in z gives the claimed relation.

1.3.2 Schur functions

Denote by $\mathcal{U}(N)$ the (compact Lie) group of all the unitary matrices[1] of size N. A (finite-dimensional) *representation* of $\mathcal{U}(N)$ is a continuous map

$$T\colon \mathcal{U}(N) \to GL(m, \mathbb{C})$$

(for some $m = 1, 2, \ldots$) which respects the group structure: $T(UV) = T(U)T(V)$, $U, V \in \mathcal{U}(N)$. A representation is called *irreducible* if it has no *invariant* subspaces $E \subset \mathbb{C}^m$ ($E \neq 0$ or $\mathbb{C}^m$), i.e., such that $T(\mathcal{U}(N))E \subset E$.

The classification of irreducible representations of $\mathcal{U}(N)$ (equivalently, of $GL(N, \mathbb{C})$ by analytic continuation — "unitary trick" of H. Weyl) is one of high points of the classical representation theory. It is due to Hermann Weyl in mid-1920's. In order to understand how it works, let us restrict T to the abelian subgroup of diagonal unitary

[1] $U^* = U^{-1}$, where U^* is the conjugate transpose.

matrices

$$\mathcal{H}_N := \left\{ \mathrm{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N}) : \varphi_1, \ldots, \varphi_N \in \mathbb{R} \right\}.$$

Any commuting family of (diagonalizable[2]) matrices can be simultaneously diagonalized. In particular, this is true for $T(\mathcal{H}_N)$. Hence, for $1 \le j \le m$,

$$\mathbb{C}^m = \bigoplus_{j=1}^{m} \mathbb{C} v_j, \qquad T\big(\,\mathrm{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N})\big) v_j = t_j(e^{i\varphi_1}, \ldots, e^{i\varphi_N}) \cdot v_j,$$

where each t_j is a continuous homomorphism $\mathcal{H}_N \to \mathbb{C}$. Any such homomorphism has the form

$$t(z_1, \ldots, z_N) = z_1^{k_1} \cdots z_N^{k_N}, \qquad k_1, \ldots, k_N \in \mathbb{Z}.$$

Each N-tuple $(k_1, \ldots, k_N) \in \mathbb{Z}^N$ for $t = t_j$, $1 \le j \le m$, is called a *weight* of the representation T. There is a total of m weights (which is the dimension of the representation).

Theorem 1.5. *Irreducible representations of $\mathcal{U}(N)$ are in one-to-one correspondence with ordered N-tuples $\lambda = (\lambda_1 \ge \cdots \ge \lambda_N) \in \mathbb{Z}^N$.*

The correspondence is established by requiring that λ is the unique highest (in lexicographic order) weight of the corresponding representation. Then the generating function of all weights of this representation T_λ can be written as

$$\begin{aligned} &\mathrm{Trace}\left(T_\lambda\left(\mathrm{diag}(z_1, \ldots, z_N)\right)\right) \\ &= \sum_{(k_1, \ldots, k_N) \text{ weight of } T_\lambda} z_1^{k_1} \cdots z_N^{k_N} = \frac{\det\left[z_i^{N+\lambda_j-j}\right]_{i,j=1}^{N}}{\det\left[z_i^{N-j}\right]_{i,j=1}^{N}}. \end{aligned} \tag{1.14}$$

Note that the denominator in (1.14) is the Vandermonde determinant which evaluates to

$$\det\left[z_i^{N-j}\right]_{i,j=1}^{N} = \prod_{1 \le i < j \le N} (z_i - z_j).$$

The numerator in (1.14) is necessarily divisible by the denominator because of its skew-symmetry with respect to $z_i \leftrightarrow z_j$, and thus the ratio is a finite linear combination of the monomials of the form $z_1^{k_1} \cdots z_N^{k_N}$, $k_1, \ldots, k_N \in \mathbb{Z}$ (i.e., an element of $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]^{S(N)}$).

There exists another basis of Λ which is more useful for our goals and it is defined in terms of Schur functions. We start by defining the Schur polynomials.

Definition 1.9 (Schur polynomial (1.14))**.** *The Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ is a symmetric polynomial in N variables $x_1, \ldots, x_N$ parametrized by $\lambda \in \mathcal{Y}$ with $\ell(\lambda) \le N$,*

[2] Any finite-dimensional representation of a finite or compact group, in particular, $U(N)$, is unitary in a suitable basis, hence all our matrices are diagonalizable.

defined by

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(x_i-x_j)}. \tag{1.15}$$

Definition 1.10 (Vandermonde determinant)**.**

$$\Delta_N(x) = \prod_{1\le i<j\le N}(x_i-x_j) = \det\left[x_i^{N-j}\right]_{1\le i,j\le N}. \tag{1.16}$$

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N}}{\det\left[x_i^{N-j}\right]_{1\le i,j\le N}}.$$

Let us verify that (1.15) has the claimed properties. First of all,

$$\det\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N}$$

is a polynomial with zeroes whenever $x_i = x_j$ for $i \neq j$. Thus this determinant is **divisible** by $x_i - x_j$ for all $i \neq j$. Therefore $s_\lambda(x_1,\dots,x_N)$ is a polynomial. Furthermore, both numerator and denominator are antisymmetric over transpositions of the x_i's. Thus the Schur polynomial are symmetric as claimed. Here is a first property of the Schur polynomials.

$$\left[x_i^{N-j}\right]_{1\le i,j\le N} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{bmatrix}$$

$$\left[x_i^{\lambda_j+N-j}\right]_{1\le i,j\le N} = \begin{bmatrix} x_1^{\lambda_N} & x_1^{1+\lambda_{N-1}} & x_1^{2+\lambda_{N-2}} & \cdots & x_1^{N-1+\lambda_1} \\ x_2^{\lambda_N} & x_2^{1+\lambda_{N-1}} & x_2^{2+\lambda_{N-2}} & \cdots & x_2^{N-1+\lambda_1} \\ x_3^{\lambda_N} & x_3^{1+\lambda_{N-1}} & x_3^{2+\lambda_{N-2}} & \cdots & x_3^{N-1+\lambda_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_N^{\lambda_N} & x_N^{1+\lambda_{N-1}} & x_N^{2+\lambda_{N-2}} & \cdots & x_N^{N-1+\lambda_1} \end{bmatrix}$$

Lemma 1.1. *Let $\ell(\lambda) \le N$. Then*

$$\pi_{N+1}s_\lambda(x_1,\dots,x_{N+1}) := s_\lambda(x_1,\dots,x_N,0) = s_\lambda(x_1,\dots,x_N) \tag{1.17}$$

and

$$\pi_{\ell(\lambda)}s_\lambda(x_1,\dots,x_{\ell(\lambda)}) = 0. \tag{1.18}$$

Proof 1.4. • *If $\ell(\lambda) \leq N$, then $\lambda_{N+1} = 0$, so*

$$
\begin{aligned}
s_\lambda(x_1, \ldots, x_N, 0) &= \frac{\det\left[x_i^{\lambda_j+N+1-j}\right]_{1\leq i,j\leq N+1}}{\det\left[x_i^{N+1-j}\right]_{1\leq i,j\leq N+1}} \\
&= \frac{\det\begin{bmatrix} x_1^{\lambda_1+N} & \cdots & x_1^{\lambda_N+1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_N^{\lambda_1+N} & \cdots & x_N^{\lambda_N+1} & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}{\det\begin{bmatrix} x_1^{N} & \cdots & x_1^{1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_N^{N} & \cdots & x_N^{1} & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}} \\
&= \frac{\det\begin{bmatrix} x_1^{\lambda_1+N} & \cdots & x_1^{\lambda_N+1} \\ \vdots & \ddots & \vdots \\ x_N^{\lambda_1+N} & \cdots & x_N^{\lambda_N+1} \end{bmatrix}}{\det\begin{bmatrix} x_1^{N} & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ x_N^{N} & \cdots & x_N \end{bmatrix}} \\
&= \frac{\det\begin{bmatrix} x_1^{\lambda_1+N-1} & \cdots & x_1^{\lambda_N} \\ \vdots & \ddots & \vdots \\ x_N^{\lambda_1+N-1} & \cdots & x_N^{\lambda_N} \end{bmatrix} \cdot x_1 x_2 \cdots x_N}{\det\begin{bmatrix} x_1^{N-1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ x_N^{N-1} & \cdots & 1 \end{bmatrix} \cdot x_1 x_2 \cdots x_N} \\
&= s_\lambda(x_1, \ldots, x_N).
\end{aligned}
\tag{1.19}
$$

This proves (1.17).

- *If $\ell(\lambda) = N$, then $\lambda_N \geq 1$, $\lambda_{N+1} = 0$, so*

$$
\begin{aligned}
\pi_N s_\lambda(x_1, \ldots, x_N) &= s_\lambda(x_1, \ldots, x_{N-1}, 0) \\
&= \Delta_N(x)^{-1} \det\begin{bmatrix} x_1^{\lambda_1+N-1} & \cdots & x_1^{\lambda_N} \\ \vdots & \ddots & \vdots \\ x_{N-1}^{\lambda_1+N-1} & \cdots & x_{N-1}^{\lambda_N} \\ 0 & \cdots & 0 \end{bmatrix} \\
&= 0,
\end{aligned}
\tag{1.20}
$$

which proves (1.18).

As a consequence of this theorem, the sequence of symmetric polynomials $s_\lambda(x_1, \dots, x_N)$ with λ fixed and varying number of variables $N \geq \ell(\lambda)$, defines an element of Λ. This element is called Schur symmetric function s_λ, where we set $s_\varnothing(x) = 1$.

An important property is that $\{s_\lambda, \lambda \in \mathcal{Y}\}$ forms a basis of Λ.

Proposition 1.1 (Jacobi-Trudi formulas)**.** *The Schur functions $\{s_\lambda, \lambda \in \mathbb{Y}\}$ forms a basis of Λ. Their relation with the generators e_k, h_k are given by the Jacobi-Trudi formulas*

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{1 \leq i,j \leq \ell(\lambda)} = \det[e_{\lambda'_i - i + j}]_{1 \leq i,j \leq \ell(\lambda')}, \tag{1.21}$$

where by definition $h_k \equiv e_k \equiv 0$, for $k < 0$.

Proof 1.5. *The proof of the fact that Schur functions form a linear basis of Λ can be found in [8] Chap. 1, Sect. 3.*

Let us prove only the first identity, namely $s_\lambda = \det[h_{\lambda_i - i + j}]$. Let $n = \ell(\lambda)$ and define $e_r^{(k)} = e_r(x_1, \dots, \hat{x}_k, \dots, x_n)$ for $1 \leq k, r \leq n$. Then

$$E^{(k)}(z) := \sum_{r=0}^{n-1} e_r^{(k)} z^r = \prod_{i \neq k} (1 + x_i z).$$

Thus we have

$$H(z) E^{(k)}(-z) = 1/(1 - x_k z) = 1 + x_k z + x_k^2 z^2 + \cdots$$

Comparing the coefficients of z^m we have the identity

$$\sum_{j=1}^{n} (-1)^{n-j} h_{m-n+j} e_{n-j}^{(k)} = x_k^m.$$

For given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we define

$$H_\alpha = [h_{\alpha_i - n + j}]_{1 \leq i,j \leq n}, \quad M = [(-1)^{n-j} e_{n-j}^{(k)}]_{1 \leq j,k \leq n}.$$

Then, $H_\alpha M = [x_j^{\alpha_i}]_{1 \leq i,j \leq n}$. Consequently

$$\det[H_\alpha] \det[M] = \det[x_j^{\alpha_i}].$$

By choosing $\alpha = \delta := (n-1, n-2, \dots, 0)$ we get

$$\det[x_j^{n-i}] = \Delta_n(x) = \det[M] \det[h_{j-i}] = \det[M]. \tag{1.22}$$

By choosing $\alpha = \delta + \lambda = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n)$ *we obtain*

$$s_\lambda = \frac{\det[x_j^{\alpha_i}]}{\det[x_j^{n-i}]} = \det[H_{\lambda+\delta}] = \det[h_{\lambda_i - i + j}]$$

as claimed.

The proof of the second identity can be found in [8] Chap. 1, Sect. 2 (pages 22-23). The idea is the following. Define matrices $H = [h_{i-j}]_{0 \le i,j \le N}$ *and*

$$E = [(-1)^{i-j} e_{i-j}]_{0 \le i,j \le N}, \quad N = \ell(\lambda)' - 1.$$

Using the identity obtained by comparing the coefficients of z^n *in the relation* $H(z)E(-z) = 1$ *one gets that* $HE = 1$ *and* $\det[E] = \det[H] = 1$*. Then the determinant of a submatrix, which is equal to the determinant of the complementary cofactor of* E^T*. Some computations shows that the latter is given by the last formula in (1.21).*

Remark 1.5. *From (1.22), we have* $\det[h_{j-i}] = 1$ *by assumption:* $h_0 = 1$ *and* $h_k = 0$ *for all* $k < 0$*.*

Now consider a measure giving to a configuration $\lambda \in \mathcal{Y}$ a weight $s_\lambda(x) \cdot s_\lambda(y)$, where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ are two sets of variables. If we want to turn this weight into a probability measure, we need

1. positive weights, $s_\lambda(x_1, x_2, \ldots) \ge 0$, for any $\lambda \in \mathcal{Y}$;

2. a normalization constant, i.e. to compute $\sum_{\lambda \in \mathcal{Y}} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots)$.

Theorem 1.6 (Cauchy identities)**.** *The following identities hold:*

1.
$$\sum_{\lambda \in Y} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j \ge 1} \frac{1}{1 - x_i y_j}, \tag{1.23}$$

2.
$$\begin{aligned} \frac{\sum_{\lambda \in Y} p_\lambda(x_1, x_2, \ldots) p_\lambda(y_1, y_2, \ldots)}{Z_\lambda} &= \exp\left(\sum_{k \ge 1} \frac{p_k(x_1, x_2, \ldots) p_k(y_1, y_2, \ldots)}{k}\right) \\ &= \prod_{i,j \ge 1} \frac{1}{1 - x_i y_j}, \end{aligned} \tag{1.24}$$

where $p_\lambda := p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_{\ell(\lambda)}}$ *and* $Z_\lambda := \prod_{i \ge 1} (i^{m_i} m_i!)$*,*

Remark 1.6. *The RHS of (1.23) should be viewed as formal power series using*

$$\frac{1}{1 - x_i y_j} = 1 + x_i y_j + (x_i y_j)^2 + \cdots .$$

In order to prove the Cauchy Identity, we have to antepone the proof of another identity, from which (1.23) will easily follow:

Theorem 1.7 (Cauchy-Binet Identity)**.**

$$\begin{aligned}&\det\left[\int_\Omega d\omega(x)\Phi_i(x)\Psi_j(x)\right]_{1\le i,j\le N}\\&\quad=\frac{1}{N!}\int_{\Omega^N} d\omega(x_1)\dots d\omega(x_N)\det[\Phi_i(x_j)]_{1\le i,j\le N}\det[\Psi_i(x_j)]_{1\le i,j\le N}.\end{aligned}\tag{1.25}$$

Proof 1.6. *By multi-linearity of the determinant we get*

$$\begin{aligned}\det\left[\int_\Omega dw(x)\Phi_i(x)\Psi_j(x)\right]_{1\le i,j\le N}&=\int_{\Omega^N} dw(x_1)\cdot\ldots\cdot dw(x_N)\det\left[\Phi_i(x_i)\Psi_j(x_i)\right]_{1\le i,j\le N}\\&=\int_{\Omega^N} dw(x_1)\cdot\ldots\cdot dw(x_N)\prod_{i=1}^{n}\Phi_i(x_i)\det\left[\Psi_j(x_i)\right]_{1\le i,j\le N}.\end{aligned}\tag{1.26}$$

Renaming $x_i=y_{\sigma(i)}, i=1,\dots,n,$

$$\begin{aligned}(1.26)&=\int_{\Omega^N} dw(y_{\sigma(1)})\cdot\ldots\cdot dw(y_{\sigma(N)})\prod_{i=1}^{N}\Phi_i(y_{\sigma(i)})\det\left[\Psi_j(y_{\sigma(i)})\right]_{1\le i,j\le N}\\&=\int_{\Omega^N} dw(y_{\sigma(1)})\cdot\ldots\cdot dw(y_{\sigma(N)})\prod_{i=1}^{N}\Phi_i(y_{\sigma(i)})(-1)^{|\sigma|}\det\left[\Psi_j(y_i)\right]_{1\le i,j\le N}.\end{aligned}\tag{1.27}$$

Since the integral does not depend on σ*,*

$$\begin{aligned}(1.27)&=\frac{1}{N!}\sum_{\sigma\in S_N}\int_{\Omega^N} dw(y_1)\cdot\ldots\cdot dw(y_N)\prod_{i=1}^{N}\Phi_i(y_{\sigma(i)})(-1)^{|\sigma|}\det\left[\Psi_j(y_i)\right]_{1\le i,j\le N}\\&=\frac{1}{N!}\int_{\Omega^N} dw(y_1)\cdot\ldots\cdot dw(y_N)\left(\sum_{\sigma\in S_N}(-1)^{|\sigma|}\prod_{i=1}^{N}\Phi_i(y_{\sigma(i)})\right)\det\left[\Psi_j(y_i)\right]_{1\le i,j\le N}.\end{aligned}\tag{1.28}$$

We have finished since $\sum_{\sigma\in S_N}(-1)^{|\sigma|}\prod_{i=1}^{N}\Phi_i(y_{\sigma(i)})=\det\left[\Phi_i(y_j)\right]_{1\le i,j\le N}$*.*

Proof 1.7. *Proof of Theorem 1.6.*

Consider $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$. Then

$$
\begin{aligned}
&\Delta_N(\vec{x})\Delta_N(\vec{y}) \sum_{\lambda \in Y} s_\lambda(\vec{x}) s_\lambda(\vec{y}) \\
&\overset{def}{=} \sum_{\lambda_1 > \lambda_2 > \dots > \lambda_n \geq 0} \det \left[x_i^{\lambda_j + N - j} \right]_{1 \leq i,j \leq N} \det \left[y_i^{\lambda_j + N - j} \right]_{1 \leq i,j \leq N} \\
&\overset{\xi_j = \lambda_j + N - j}{=} \sum_{\xi_1 > \xi_2 > \dots > \xi_N \geq 0} \det \left[x_i^{\xi_j} \right]_{1 \leq i,j \leq N} \det \left[y_i^{\xi_j} \right]_{1 \leq i,j \leq N} \\
&= \frac{1}{N!} \sum_{\xi_1, \xi_2, \dots, \xi_N \geq 0} \det \left[x_i^{\xi_j} \right]_{1 \leq i,j \leq N} \det \left[y_i^{\xi_j} \right]_{1 \leq i,j \leq N},
\end{aligned}
\tag{1.29}
$$

where the last equality follows because of the summmand is zero whenever $\xi_i = \xi_j$ for $i \neq j$ and it is symmetric. Applying the Cauchy-Binet identity (1.25) we obtain

$$
\begin{aligned}
(1.29) = \det \left[\sum_{\xi > 0} (x_i y_j)^\xi \right]_{1 \leq i,j \leq N} &= \det \left[\frac{1}{1 - x_i y_j} \right]_{1 \leq i,j \leq N} \\
= \prod_{i,j=1}^N \frac{1}{1 - x_i y_j},
\end{aligned}
\tag{1.30}
$$

where the last step is proved in the following theorem.

Theorem 1.8. *It holds*

$$
\left(\prod_{1 \leq i < j \leq N} (x_i - x_j)(y_i - y_j) \right)^{-1} \det \left[\frac{1}{1 - x_i y_j} \right] = \prod_{i,j=1}^N \frac{1}{1 - x_i y_j}.
$$

Proof 1.8. *We won't give a complete proof of this identity, but we will illustrate a direct computation for $N = 3$. The same ideas can be generalized for any N.*

$$
\begin{aligned}
\det \begin{bmatrix} \frac{1}{1-x_1y_1} & \frac{1}{1-x_1y_2} & \frac{1}{1-x_1y_3} \\ \frac{1}{1-x_2y_1} & \frac{1}{1-x_2y_2} & \frac{1}{1-x_2y_3} \\ \frac{1}{1-x_3y_1} & \frac{1}{1-x_3y_2} & \frac{1}{1-x_3y_3} \end{bmatrix} &= \prod_{i,j=1}^3 \frac{1}{1 - x_i y_j} \\
\times \det &\begin{bmatrix} (1-x_1y_2)(1-x_1y_3) & (1-x_1y_1)(1-x_1y_3) & (1-x_1y_1)(1-x_1y_2) \\ (1-x_2y_2)(1-x_2y_3) & (1-x_2y_1)(1-x_2y_3) & (1-x_2y_1)(1-x_2y_2) \\ (1-x_3y_2)(1-x_3y_3) & (1-x_3y_1)(1-x_3y_3) & (1-x_3y_1)(1-x_3y_2) \end{bmatrix}
\end{aligned}
\tag{1.31}
$$

We want to prove that the determinant in the last line of (1.31) coincides with the Vandermonde determinant. Let C_i indicate the i-th column of the matrix. If we subtract C_1 to C_2 and C_3,

$$\det \begin{bmatrix} (1-x_1y_2)(1-x_1y_3) & (1-x_1y_1)(1-x_1y_3) & (1-x_1y_1)(1-x_1y_2) \\ (1-x_2y_2)(1-x_2y_3) & (1-x_2y_1)(1-x_2y_3) & (1-x_2y_1)(1-x_2y_2) \\ (1-x_3y_2)(1-x_3y_3) & (1-x_3y_1)(1-x_3y_3) & (1-x_3y_1)(1-x_3y_2) \end{bmatrix}$$

$$\overset{C_2\to C_2-C_1}{=} \quad \overset{C_3\to C_3-C_1}{=} \det \begin{bmatrix} (1-x_1y_2)(1-x_1y_3) & (1-x_1y_3)(y_2-y_1)x_1 & (1-x_1y_1)(y_3-y_1)x_1 \\ (1-x_2y_2)(1-x_2y_3) & (1-x_2y_3)(y_2-y_1)x_2 & (1-x_2y_1)(y_3-y_1)x_2 \\ (1-x_3y_2)(1-x_3y_3) & (1-x_3y_3)(y_2-y_1)x_3 & (1-x_3y_1)(y_3-y_1)x_3 \end{bmatrix}$$

$$\overset{linearity}{=} (y_3-y_1)(y_2-y_1)\det \begin{bmatrix} (1-x_1y_2)(1-x_1y_3) & (1-x_1y_3)x_1 & (1-x_1y_1)x_1 \\ (1-x_2y_2)(1-x_2y_3) & (1-x_2y_3)x_2 & (1-x_2y_1)x_2 \\ (1-x_3y_2)(1-x_3y_3) & (1-x_3y_3)x_3 & (1-x_3y_1)x_3 \end{bmatrix} \tag{1.32}$$

Now subtracting C_2 to C_3,

$$\overset{C_3\to C_3-C_2}{=} (y_3-y_1)(y_2-y_1)(y_3-y_2)\det \begin{bmatrix} 1-(y_2+y_3)x_1+y_2y_3x_1^2 & (1-x_1y_3)x_1 & (y_3-y_1)x_1^2 \\ 1-(y_2+y_3)x_2+y_2y_3x_2^2 & (1-x_2y_3)x_2 & (y_3-y_1)x_2^2 \\ 1-(y_2+y_3)x_3+y_2y_3x_3^2 & (1-x_3y_3)x_3 & (y_3-y_1)x_3^2 \end{bmatrix}$$

$$\overset{lin.}{=} (y_3-y_1)(y_2-y_1)(y_3-y_2)\det \begin{bmatrix} 1-(y_2+y_3)x_1+y_2y_3x_1^2 & x_1-y_3x_1^2 & x_1^2 \\ 1-(y_2+y_3)x_2+y_2y_3x_2^2 & x_2-y_3x_2^2 & x_2^2 \\ 1-(y_2+y_3)x_3+y_2y_3x_3^2 & x_3-y_3x_3^2 & x_3^2 \end{bmatrix}$$

$$\overset{lin.}{=} \Delta(\vec{y})\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = \Delta(\vec{y})\Delta(\vec{x}). \tag{1.33}$$

1.3.3 Skew Schur functions

Consider two sets of variable $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and let (x, y) be the union of the sets. Consider a function $f(x, y)$ symmetric w.r.t all the permutations of the x_i, y_i's. Since $f(x, y)$ is symmetric in the x_i's and in the y_i's it must be expressed in the form $f(x, y) = \sum_j f_j(x)g_j(y)$, for some symmetric functions f_j, g_j. For instance, if one takes for $f(x, y)$ the power sum. Then

$$p_k(x, y) = \sum_{i\geq 1} x_i^k + \sum_{i\geq 1} y_i^k = p_k(x) + p_k(y).$$

We want to understand how the Schur function on the union of two sets of variables, $s_\lambda(x, y)$, decompose.

Definition 1.11 (skew Schur functions)**.** *Let λ be a Young tableau. Then*

$$s_\lambda(x, y) = \sum_{\mu \in Y} s_{\lambda/\mu}(x) s_\mu(y),$$

where the coefficients $s_{\lambda/\mu}(x)$ are called skew Schur functions and are symmetric functions in x.

We study some properties of the skew Schur functions.

Proposition 1.2 (Consistency)**.** *Let x, y be two sets of variables. Then for any $\lambda, \mu \in Y$*

$$s_{\lambda/\mu}(x, y) = \sum_{\nu \in Y} s_{\lambda/\nu}(x) s_{\nu/\mu}(y). \tag{1.34}$$

Proof 1.9. *Consider three sets of variables x, y, z. Then, by definition*

$$s_\lambda(x, y, z) = \sum_{\mu \in Y} s_{\lambda/\mu}(x, y) s_\mu(z)$$

but also

$$\begin{aligned} s_\lambda(x, y, z) &= \sum_{\nu \in Y} s_{\lambda/\nu}(x) s_\nu(y, z) \\ &= \sum_{\mu \in Y} \sum_{\nu \in Y} s_{\lambda/\nu}(x) s_{\nu/\mu}(y) s_\mu(z) \end{aligned} \tag{1.35}$$

Since the equality holds for any choice of variables z, we need to have the equivalence of the coefficients of $s_\mu(z)$, which is (1.34).

References

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