1. Let $A_1, A_2, \ldots, A_m$ be subsets of an n-element set, such that

- the size of each A_i, with $1 \le i \le m$, is not divisible by 6, and
- the size of any $A_i \cap A_j$, with $1 \le i < j \le m$, is divisible by 6.

Show that $m \le 2n$.

Proof. Without loss of generality, suppose that $A_1, A_2, \ldots, A_{m_1}, A_{m_1+1}, \ldots, A_m$ are ordered such that

(1) $|A_i|$, for each $1 \le i \le m_1$, is not divisible by 2; and

(2) $|A_i|$, for each $m_1 + 1 \le i \le m$, is not divisible by 3.

Since the size of any $A_i \cap A_j$, with $1 \le i < j \le m$, is divisible by 6, we obtain

(3) $|A_i \cap A_j|$, for each $1 \le i < j \le m_1$, is divisible by 2; and

(4) $|A_i \cap A_j|$, for each $m_1 + 1 \le i < j \le m$, is divisible by 3.

Imitating the proof of the Oddtown Theorem over $\mathbb{F}_2$ (using (1) and (3)) and $\mathbb{F}_3$ (using (2) and (4)), respectively, we obtain $m_1 \le n$ and $m - m_1 \le n$. So $m = m_1 + (m - m_1) \le 2n$. ∎

odd town , n citizens

no more than n clubs can be formed

1. $\forall i$, size of A_i is odd : $2 \nmid$ card

2. $\forall i \ne j$, size of $A_i \cap A_j$ is even : $2 \mid$ card

so from (1) and (3) is the same to odd town, so $m_1 < n$ over $\mathbb{Z}_2$

6 have two factor 2, 3, so $6 \nmid |A_i|$ implies $2 \nmid |A_i|$ or $3 \nmid |A_i|$

$\forall i, j$, we have $6 \nmid |A_i \cap A_j|$

Fact Let $B_1, B_2, \ldots, B_k$ be subsets of an n-set.

If

- none of $|B_i|$, $1 \le i \le k$, is divisible by p^t;
- each $|B_i \cap B_j|$, $1 \le i < j \le k$, is divisible by p^t,

where p is a prime, Then $k \le n$.

Equivalent form of 3(b)

Light bulbs $l_1, l_2, \ldots, l_n$ are controlled by switches $s_1, s_2, \ldots, s_n$. The ith switch changes the on/off status of the ith light and possibly others, but s_i changes the status of l_j iff s_j changes the status of l_i. Initially, all the lights are off. Prove that it is possible to Turn all lights on.

2. Let $A_1, A_2, \ldots, A_m$ be distinct subsets of $\{1, 2, \ldots, n\}$ such that $|A_i \cap A_j| = \lambda$ for any $i \ne j$ and $|A_i| > \lambda$ for all i.

(a) Let x_i be a real variable associated with A_i for $i = 1, 2, \ldots, m$. Show that

$$\left(\sum_{i=1}^{m} x_i\right)^2 = \boxed{\sum_{i=1}^{m} x_i^2} + \frac{1}{\lambda}\sum_{k=1}^{n}\Big[\Big(\sum_{k\in A_i} x_i\Big)^2 - \sum_{k\in A_i} x_i^2\Big]. \qquad (0)$$

(b) Use (a) to prove that $m \le n$.

Proof. (a) We claim that for any fixed k with $1 \le k \le n$,

$$\boxed{\Big(\sum_{k\in A_i} x_i\Big)^2 - \sum_{k\in A_i} x_i^2 = 2\sum_{k\in A_i\cap A_j} x_i x_j.} \qquad (1)$$

To justify this, let $k \in A_{i_j}$ for $1 \le j \le t$. Then

$$\text{LHS of (1)} = \Big(\sum_{j=1}^{t} x_{i_j}\Big)^2 - \sum_{j=1}^{t} x_{i_j}^2 = 2\sum_{1\le p<q\le t} x_{i_p} x_{i_q} = \text{RHS of (1)}.$$

Thus the desired statement is equivalent to

$$\boxed{\lambda \sum_{1\le i<j\le m} x_i x_j = \sum_{k=1}^{n}\sum_{k\in A_i\cap A_j} x_i x_j} \qquad (2)$$

$|A_i \cap A_j| = \lambda$

let k from 1 to n

$\lambda x_i x_j$

Since $A_i \cap A_j$ contains λ elements and for each element $k \in A_i \cap A_j$, the term $x_i x_j$ appears once on the RHS of (2), in total $x_i x_j$ appears precisely $|A_i \cap A_j| = \lambda$ times. So (2) holds.

(b) Suppose the contrary: $m > n$. Then there exists a nonzero solution to the following system

theorem : nonuniform Fisher inequality

Let $C_1, C_2, \ldots, C_m$ be distinct subsets of a set of size n, such that $\forall i \ne j$, $|C_i \cap C_j| = \lambda$ where λ is a fixed constant with $1 \le \lambda \le n$

then $m \le n$

$\frac{1}{\lambda}\sum_{k=1}^{n} 2 \cdot \sum_{k\in A_i\cap A_j} x_i x_j$ RHS of 0

LHS of (0), $2 \sum_{1\le i<j\le m} x_i x_j$

here $\sum_{k=1}^{n} \sum_{k\in A_i\cap A_j} x_i x_j$

k from 1 to n, now means k appears n time, actually the maximum is λ, since $|A_i \cap A_j| = \lambda$

example, suppose k only contained in A_2, A_5, A_7, then

$\left(\sum_{k\in A_i} x_i\right)^2 = (x_2 + x_5 + x_7)^2$, $\quad \sum_{k\in A_i} x_i^2 = x_2^2 + x_5^2 + x_7^2$

$$\boxed{\sum_{k\in A_i} x_i = 0 \quad \text{for } k = 1, 2, \ldots, n.}$$

So

$$\begin{aligned}
\Big(\sum_{i=1}^{m} x_i\Big)^2 &= \sum_{i=1}^{m} x_i^2 + \frac{1}{\lambda}\sum_{k=1}^{n}\Big[\Big(\sum_{k\in A_i} x_i\Big)^2 - \sum_{k\in A_i} x_i^2\Big] \\
&= \sum_{i=1}^{m} x_i^2 - \frac{1}{\lambda}\underbrace{\sum_{k=1}^{n}\sum_{k\in A_i}}_{\to |A_i|} x_i^2 \\
&= \sum_{i=1}^{m} x_i^2 - \frac{1}{\lambda}\sum_{i=1}^{m} |A_i|\, x_i^2 \\
&= \sum_{i=1}^{m}\Big(1 - \frac{|A_i|}{\lambda}\Big) x_i^2 < 0 \quad \text{as} \quad |A_i| > \lambda \text{ for all } i,
\end{aligned}$$

while $\big(\sum_{i=1}^{m} x_i\big)^2 \geq 0$, contradicting (a). ∎

Hanshaking Theorem let $G = (V, E)$ be a graph. Then

$$\sum_{v\in V} d(v) = 2|E|.$$

hand shaking theorem

3. (a) Let $A = (a_{ij})$ be an $n \times n$ symmetric $0-1$ matrix. Show that the diagonal $a = (a_{11}, a_{22}, \ldots, a_{nn})^T$ is in the span of the columns of A over the field $\mathbb{F}_2$.

(b) Prove that the vertices of any finite simple graph can be colored red and blue so that each red vertex is adjacent to an *even* number of red vertices, and each blue vertex is adjacent to an *odd* number of red vertices.

Let $A = (v_1, v_2 \cdots\cdots v_n)$ where v_i is the ith column vector in A

WTS $a = (a_{11}, a_{22} \cdots\cdots a_{nn}) \in \operatorname{span}(v_1 \cdots\cdots v_n)$

equal to $a^T x = 0$ whenever $Ax = 0$

$a^T x = 0 \iff Ax = 0$ is equivalent to $\begin{bmatrix} A \\ a \end{bmatrix} x = 0$

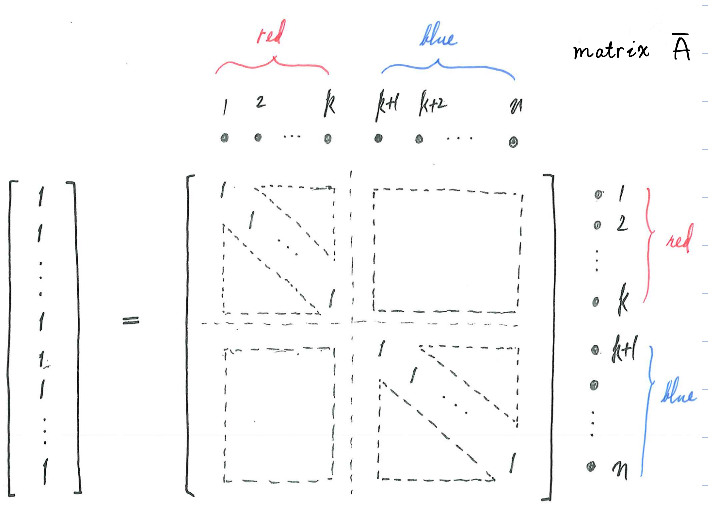
Proof. (a) To prove the statement, it suffices to prove that over $\mathbb{F}_2$ we have $a^T x = 0$ whenever $Ax = 0$. For this purpose, consider

$$x^T A x = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{1\le i<j\le n} 2a_{ij} x_i x_j. \tag{3}$$

$\mathbb{F}_2$:
1. $x^2 = x$
2. $2x = 0$

Since $x_i^2 = x_i$ over $\mathbb{F}_2$, by (3) we obtain $x^T A x = \sum_{i=1}^{n} a_{ii} x_i = a^T x$. So $a^T x = 0$ whenever $Ax = 0$. Hence the linear systems $Ax = 0$ and $\begin{bmatrix} A \\ a^T \end{bmatrix} x = 0$ have the same solution sets over $\mathbb{F}_2$. It follows that $rk(A) = rk \begin{bmatrix} A \\ a^T \end{bmatrix}$. Therefore a^T is in the span of the rows of A over $\mathbb{F}_2$. Equivalently, a is in the span of the columns of A over $\mathbb{F}_2$ as A is symmetric.

(b) Let A be the adjacency matrix of the given graph G, and let $\bar{A} = A + I$. Then $\mathbf{1} = (1, 1, \ldots, 1)^T$ is in the span of the columns of $\bar{A}$; that is, $\mathbf{1} = \sum_{j=1}^{k} \bar{a}_{i_j}$ over $\mathbb{F}_2$, where $\bar{a}_t$ is the t^{th} column of $\bar{A}$. Let us color the vertices $i_1, i_2, \ldots, i_k$ by red and all other vertices by blue. From the above equation, it is easy to see that each red vertex is adjacent to an even numbers of red vertices, and each blue vertex is adjacent to an odd number of red vertices. ∎



$\mathbb{1} = \sum_{j=1}^{k} \bar{a}_{i_j}$

here without loss of generality $\mathbb{1} = \sum_{j=1}^{k} \bar{a}_j$

binary number

4. Prove that if we drop the condition in the Graham-Pollak Theorem (see Theorem 3 in the lecture notes) that the complete bipartite graphs are edge disjoint, then $\lceil \log_2 n \rceil$ complete bipartite graphs suffice to cover K_n. Furthermore, $\lceil \log_2 n \rceil$ is also necessary.

Proof. Our proof comes in two parts; both of them involve binary expressions of integers.

Sufficiency. Label the vertices of K_n as $0, 1, \ldots, n-1$. Each i, with $0 \le i \le n-1$, can be expressed as a binary number of length $\lceil \log_2 n \rceil$ of the form $a_k a_{k-1} \ldots a_2 a_1$, where $k = \lceil \log_2 n \rceil$, $a_l \in \{0, 1\}$ for $l = 1, 2, \ldots, k$. Thus $i = \sum_{l=1}^{k} a_l 2^{l-1}$.

For $l = 1, 2, \ldots, \lceil \log_2 n \rceil$, define a complete bipartite graph $B_l = (X_l, Y_l)$ by

$$X_l = \{0 \le i \le n-1 : \text{the } l^{th} \text{ bit of } i \text{ is } 1\},$$
$$Y_l = \{0 \le i \le n-1 : \text{the } l^{th} \text{ bit of } i \text{ is } 0\}.$$

Let us show that $B_1, B_2, \ldots, B_{\lceil \log_2 n \rceil}$ cover all edges of K_n. To justify this, note that for each edges ij, with $0 \le i \ne j \le n-1$, the binary expressions of i and j must differ at some bit, say the l^{th} bit. Then ij is contained in B_l with $l \le \lceil \log_2 n \rceil$ by the definition of B_l.

Necessity. Suppose K_n is covered by complete bipartite graphs $B_1 = (X_1, Y_1)$, $B_2 = (X_2, Y_2), \ldots, B_k = (X_k, Y_k)$. To prove that $k \ge \lceil \log_2 n \rceil$, we associate with each vertex v in K_n a binary number $f(v)$ of length k, such that the i^{th} bit of $f(v)$ is equal to 1 if $v \in X_i$ and 0 otherwise. Given any pair of distinct vertices u and v in K_n, we see that the edge uv is covered by some $B_i = (X_i, Y_i)$. Thus

$$\{\text{the } i^{th} \text{ bit of } f(u), \text{the } i^{th} \text{ bit of } f(v)\} = \{1, 0\},$$

which implies $f(u) \ne f(v)$. Since the total number of binary numbers of length k is 2^k, we have $n = |\{f(v) : v \in V(K_n)\}| \le 2^k$, so $k \ge \lceil \log_2 n \rceil$, as desired. ∎

Graham - Pollak theorem

if the edge set of complete K_n is the disjoint union of the edge sets of m complete graph, then $m \ge n-1$

sufficiency: form $\lceil \log_2 n \rceil$ complete bipartite graph to cover K_n

inverse

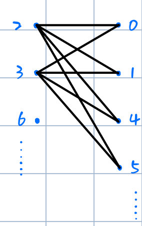
WTS $k \ge \lceil \log_2 n \rceil$

Integer i	Binary Expression	# of bits
0	0	1
1	1	1
2	10	2
3	11	2
4	100	3
5	101	3
6	110	3
⋮	⋮	⋮

$a_3 \quad a_2 \quad a_1$

$\lceil \log_2 3 \rceil = 2 > 1$

for example let $l = 2$ so we have $B_2 = (X_2, Y_2)$



B_2

Fact The number of bits in the binary expression of $k = \lceil \log_2 (k+1) \rceil$.