introductory content : sheaf

def : Spec (R)

Let R be a ring, Spec (R) is a collection of all the prime ideals in R, $\mathrm{Spec}(R) = \{ p \triangleleft R, p \text{ is prime} \}$

so if a prime ideal p in R, $p \in \mathrm{Spec}(R)$, renote as $[p]$

def : category

a category $\mathscr{C}$ consists of a collection of objects $Ob(\mathscr{C})$; and for two objects $A, B \in Ob(\mathscr{C})$, a set $\mathcal{M}or(A, B)$ called the set of morphisms of A into B; and for three objects A, B, D **satisfying** a law of composition

$$\mathcal{M}or(B, D) \times \mathcal{M}or(A, B) = \mathcal{M}or(A, D)$$

satisfying the following axioms

1. two set $\mathcal{M}or(A, B)$ and $\mathcal{M}or(A', B')$ are disjoint unless $A = A'$ and $B = B'$, in which they are equal

2. for each object A of $\mathscr{C}$, there is a morphism $id_A \in \mathcal{M}or(A, A)$ which act as left and right identity for the elements of $\mathcal{M}or(A, B)$ and $\mathcal{M}or(B, A)$ respectively, $\forall B \in Ob(\mathscr{C})$

3. law of composition is associative, Given $f \in \mathcal{M}or(A, B)$, $g \in \mathcal{M}or(B, D)$ and $h \in \mathcal{M}or(D, E)$ $\quad h \circ (g \circ f) = (h \circ g) \circ f$

def: functor

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A functor $F: \mathcal{C} \to \mathcal{C}'$ consist of the following data

1. a map between objects: $F: Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{C}')$

2. for every A, B in $Ob(\mathcal{C})$, a map between morphisms

$$F: \mathrm{Mor}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathcal{C}'}(F(A), F(B))$$

satisfying (1). for $A \in Ob(\mathcal{C})$, we have $F(id_A) = id_{F(A)}$

(2): $\forall A, B, D \in Ob(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B,D)$,

we have $F(f \circ g) = F(f) \circ F(g)$

But if the functor F is **contravariant**, means

$$F: \mathrm{Mor}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Mor}_{\mathcal{C}'}(F(B), F(A))$$

$\forall A, B, D \in Ob(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B,D)$, we have

$$F(f \circ g) = F(g) \circ F(f)$$

def: natural transformation

Let $F, G: \mathcal{C} \longrightarrow \mathcal{C}'$ be two functors. A natural transformation $\eta: F \longrightarrow G$ is a family of morphisms

$$\eta_A \in \mathrm{Mor}_{\mathcal{C}'}(F(A), G(A))$$

$A \in Ob(\mathcal{C})$, satisfying the following diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"] & G(B)
\end{tikzcd}$$

natural transformation

def: R - module

Let R be a ring, and M is an abelian group, if there is a map

$$\begin{aligned} R \times M &\longrightarrow M \\ (a, x) &\longmapsto ax \end{aligned}$$

and satisfying the following properties

1. $a(x+y) = ax + ay$
2. $(a+b)x = ax + bx$ where $x, y \in M$
3. $1x = x$ $\quad a, b, 1 \in R$
4. $(ab)x = a(bx)$

we call M is a left R-module

def: limit and colimit in category theory

Let $(I, \leq)$ is a partially ordered set, $\forall \lambda, \mu \in I$, always exists $\gamma \in I$ s.t. $\lambda < \gamma$, $\mu < \gamma$, so I is called direct set

J is a category, where $Ob(J) = I$, $\forall \lambda, \mu \in I$, the morphism is $Mor(\lambda, \mu) = \emptyset$, when $\lambda \nleq \mu$; and $Mor(\lambda, \mu) =$ singleton set

if $\mathcal{M}_R$ is a R module category, so functor $F: J \longrightarrow \mathcal{M}_R$ defines a family of R-module M_λ

$$\begin{aligned} J &\longrightarrow \mathcal{M}_R \\ \lambda &\longmapsto M_\lambda \end{aligned}$$

and a family of morphisms $f_{\mu\lambda}$

$$f_{\mu\lambda} \in Mor_R(M_\lambda, M_\mu)$$

1. $\forall \lambda \in I$, $f_{\lambda\lambda} = id_{M_\lambda}$

2. $\forall \lambda \leq \mu \leq \nu$, $f_{\nu\mu} \circ f_{\mu\lambda} = f_{\nu\lambda}$

$$\begin{tikzcd}
\lambda \arrow[r] & M_\lambda \arrow[d, "f_{\mu\lambda}"] \\
\mu \arrow[r] & M_\mu \arrow[d, "f_{\nu\mu}"] \\
\nu \arrow[r] & M_\nu
\end{tikzcd}$$
($\lambda \leq \mu \leq \nu$)

so $\{M_\lambda, f_{\mu\lambda}\}$ is a direct system

so if we have a direct system $\{M_\lambda, f_{\mu\lambda}\}$, and there exists R module M and homomorphisms $\{\varphi_\lambda : M_\lambda \longrightarrow M\}$ satisfying

1. $\forall \lambda \leq \mu$, $\varphi_\lambda = \varphi_\mu \cdot f_{\mu\lambda}$

$$\begin{tikzcd}
\lambda \arrow[r, "F"] & M_\lambda \arrow[d, "f_{\mu\lambda}"] \arrow[dr, "\varphi_\lambda"] & \\
\mu \arrow[r, "F"] & M_\mu \arrow[r, "\varphi_\mu"'] & M
\end{tikzcd}$$

2. if there is a R module N and $\{\psi_\lambda : M_\lambda \longrightarrow N\}$ homomorphism $\forall \lambda \leq \mu$: $\psi_\lambda = \psi_\mu \cdot f_{\mu\lambda}$, then $\exists!$ R module homomorphism $g : M \to N$

$$\begin{tikzcd}
 & & M \arrow[dd, "g", "\exists!"'] \\
M_\mu \arrow[urr, bend left, "\varphi_\mu"] \arrow[drr, bend right, "\psi_\mu"'] & M_\lambda \arrow[l, "f_{\mu\lambda}"'] \arrow[ur, "\varphi_\lambda"] \arrow[dr, "\psi_\lambda"'] & \\
 & & N
\end{tikzcd}$$

so M is the colimit / direct limit of the direct system $\varinjlim M_\lambda = M$

def : presheaf

Let X be a topological space. A presheaf $\mathcal{F}$ on X consist

1. for all open $U \subset X$, a set $\mathcal{F}(U)$ is an Abelian group

2. whenever $U \subset V \subset X$, a map $res_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ called the restriction map, satisfying

3. $res_{UU} = id_U$

4. if $\mathcal{U} \subset V \subset W$, then $res_{V\mathcal{U}} \circ res_{WV} = res_{W\mathcal{U}}$

a presheaf is a contravariant functor

$\mathcal{U}$ in category of open subsets of X, we have a correspoundant category of set, so presheaf is regarded as a functor $\mathcal{F}$

$\forall \mathcal{U}, V \subset X$, and they are open. $Mor(\mathcal{U}, V) = \{ id_{\mathcal{U}V}, \mathcal{U} \subset V \}$

$Mor(\mathcal{U}, V) \longrightarrow Mor(\mathcal{F}(V), \mathcal{F}(\mathcal{U}))$ contravariant functor

let $\mathcal{G}$ is another presheaf, so we have natural transformation α

$\forall \mathcal{U}$ open in X

$$\begin{tikzcd}
\mathcal{F}(\mathcal{U}) \arrow[r, "\alpha_{\mathcal{U}}"] & \mathcal{G}(\mathcal{U}) \\
\mathcal{F}(V) \arrow[u, "res_{V\mathcal{U}}"] \arrow[r, "\alpha_V"] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{G}(V) \arrow[u, "res'_{V\mathcal{U}}"']
\end{tikzcd}$$

def: sheaf $\mathcal{F}(\mathcal{U})$ or renote $\Gamma(U, \mathcal{F})$

a presheaf $\mathcal{F}$ is a sheaf if for all open set $V \subset X$, and all open coverings $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of V, the two properties hold

1. (Identity axiom) If $s_1, s_2 \in \mathcal{F}(V)$ and $res_{V\mathcal{U}_\alpha}(s_1) = res_{V\mathcal{U}_\alpha}(s_2)$ in each set $\mathcal{F}(\mathcal{U}_\alpha)$, then $s_1 = s_2$ local $\longrightarrow$ global

2. (Gluability axiom) If $s_\alpha \in \mathcal{F}(\mathcal{U}_\alpha)$ is a set of elements s.t. $\forall \alpha, \beta \in I$, $res_{\mathcal{U}_\alpha\, \mathcal{U}_\alpha \cap \mathcal{U}_\beta}(s_\alpha) = res_{\mathcal{U}_\beta\, \mathcal{U}_\alpha \cap \mathcal{U}_\beta}(s_\beta)$ in $\mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ then there exists a $\gamma \in \mathcal{F}(V)$ such that $\forall \alpha : res_{V\mathcal{U}_\alpha}(\gamma) = s_\alpha$

def: stalk of sheaf at a point

If $\mathcal{F}$ is a sheaf on X and $x \in X$ we can form

$$\mathcal{F}_x = \varinjlim_{\substack{x \in U \\ \text{all open } U}} \mathcal{F}(U) = \{(s, U),\ U \text{ open in } X,\ x \in U,\ s \in \Gamma(U, \mathcal{F})\} / \sim$$

where $(s, U) \sim (t, V)$ if $\exists$ open $W \subset U$, $W \subset V$ with $x \in W$

s.t. $res_{UW}(s) = res_{VW}(t)$

$U \subset X$ regard as index category ordered by inclusion, $V \subset U$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "res_{VU}"] \arrow[dr, "\rho_{xU}"'] & \mathcal{F}(V) \arrow[r, "res_{WV}"] \arrow[d, "\rho_{xV}"] & \mathcal{F}(W) \arrow[dl, "\rho_{xW}"] \\
& \mathcal{F}_x = \coprod_{x \in U} \mathcal{F}(U) / \sim &
\end{tikzcd}$$

$p \in U, V, W$

direct limit

def: germ

according to the definition of direct limit, $\forall$ open set $U \subset X$,

$\exists$ homomorphism $\rho_{xU} : \mathcal{F}(U) \longrightarrow \mathcal{F}_x$, for $x \in V \subset U$

satisfying $\rho_{xV} \circ res_{VU} = \rho_{xU}$, for section $s \in \mathcal{F}(U)$

$\rho_{xU}(s) = s_x \in \mathcal{F}_x$. We call s_x the germ of s of the sheaf $\mathcal{F}$

at the point x

def: section

$\mathcal{U}$ is an open set in topological space and there is a sheaf $\mathcal{F}$ on X, $s \in \mathcal{F}(\mathcal{U})$ means s is a section on $\mathcal{U}$

def: structure sheaf → zero divisor

for all open set $\mathcal{U} \subset X$, $\mathcal{O}_X(\mathcal{U})$ is a ring of regular functions.

f is regular means $\forall x \in \mathcal{U}$, exists a neighbourhood V s.t. $x \in V \subset \mathcal{U}$ on set V, $f = g/h$, where g, h are two polynomials and h has not zeros on V (means f hasn't got poles on V)

def: $\mathcal{O}_X$-module sheaf

1. $\mathcal{O}_X$-module sheaf $\mathcal{F}$ is a sheaf
2. $\forall$ open set $\mathcal{U} \subset X$, $\mathcal{F}(\mathcal{U})$ is a $\mathcal{O}_X(\mathcal{U})$-module
3. $V \subset \mathcal{U}$ they are open in X: $res_{UV}: \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$

   it is module homomorphism

def: tensor product

there are two R-module M, N and their tensor $M \otimes_R N$ is a new R module

$(am) \otimes n = a(m \otimes n)$, $m \otimes (bn) = b(m \otimes n)$

where $a, b \in R$, $m \in M$, $n \in N$

two $\mathcal{O}_X$ sheaves $\mathcal{F}, \mathcal{G}$, their tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is a new sheaf, for all open set $\mathcal{U} \subset X$

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(\mathcal{U}) := \mathcal{F}(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{G}(\mathcal{U})$$

def: line bundle / invertible sheaf $\mathcal{O}_X(D)$, divisor D

a $\mathcal{O}_X$-module sheaf $\mathcal{L}$ is called invertible sheaf, if exists open coverings $\{\mathcal{U}_i\}_{i \in I}$, such that $\forall i$, on $\mathcal{U}_i$, we have $\mathcal{L}|_{\mathcal{U}_i} \cong \mathcal{O}_X|_{\mathcal{U}_i}$ locally

the invertible property: $\exists \mathcal{L}^{-1}$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$

def: skyscraper sheaf

Let X be a topological space and $x \in X$, skyscraper sheaf $\tilde{\mathcal{F}}$ open set $\mathcal{U} \subset X$, $\mathcal{A}$ is an algebraic structure (i.e. field)

$$\text{define stalk } \tilde{\mathcal{F}}_x(\mathcal{U}) = \begin{cases} \mathcal{A}, & \text{if } x \in \mathcal{U}, \quad \text{let } \mathcal{A} = \mathbb{C} \\ 0, & \text{if } x \notin \mathcal{U} \end{cases}$$

if $x \in V \subset \mathcal{U}$, $res_{\mathcal{U}V} = id_{\mathcal{A}}$, if $x \notin V$, $res = 0$ map

$\forall$ point $p \in X$,

$$\tilde{\mathcal{F}}(p) = \begin{cases} \mathbb{C}, & \text{if and only if } p = x \\ 0, & \text{if } q \neq x \end{cases}$$

topics in homological algebra

def: complex chain $\mathscr{C}_\bullet = (C_n, \partial_n)$

R module homomorphic sequence

$$\xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \cdots\cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

called R-module complex, if $\partial_{n-1}\partial_n = 0$, where ∂_n is called differential

Let $\mathscr{C}_\bullet(C_n, \partial_n)$ and $D_\bullet(D_n, \delta_n)$ to be two R module complex

if there is a R module homomorphism $\varphi_n : C_n \longrightarrow D_n$

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\ D_n \arrow[r, "\delta_n"] & D_{n-1} \end{tikzcd}$$

$$\varphi_{n-1} \circ \partial_n = \delta_n \circ \varphi_n$$

$\varphi = \{\varphi_n\}$ is a morphism from complex $C_\bullet$ to $D_\bullet$

image: $\operatorname{Im} \partial_k = \{ x \in C_{k-1} \mid \exists\, y \in C_k \text{ s.t. } x = \partial_k(y) \} = B_{k-1}$

kernal: $\operatorname{Ker} \partial_k = \{ x \in C_k \mid \partial_k(x) = 0 \} = Z_k$

want to show $B_n \lhd Z_n$ $\quad C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$

we know that $B_n \subset Z_n$ because of $\partial_n \partial_{n+1} = 0$

and C_n is a R module, so including an Abelian group structure

def: homology group

$$H_n(C) = Z_n / B_n$$

def: exact chain

Let $(C_\cdot, \partial)$ be a chain complex, we call it exact, means

$\forall n \in \mathbb{N}$, $\mathrm{Ker}(\partial_n) = \mathrm{Im}(\partial_{n+1})$, as $B_n = Z_n$

lemma: Snake lemma

given a commutative diagram

if rows are exact

$$\begin{tikzcd}
& \mathrm{Ker} f \arrow[r] \arrow[d] & \mathrm{Ker} g \arrow[r] \arrow[d] & \mathrm{Ker} h \arrow[d] \arrow[ddll, bend left, "\partial"] & \\
& A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
A/\mathrm{Im}(f) = & \mathrm{Cok} f \arrow[r] & \mathrm{Cok} g \arrow[r] & \mathrm{Cok} h &
\end{tikzcd}$$

then exists exact sequence

$$\mathrm{Ker}(f) \hookrightarrow \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{Cok}(f) \to \mathrm{Cok}(g) \twoheadrightarrow \mathrm{Cok}(h)$$

$\partial(c_1) = i^{-1} g p^{-1}(c_1)$

where $c_1 \in \mathrm{Ker}(h)$

Moreover if $A' \hookrightarrow B' \Longrightarrow \mathrm{Ker}(f) \hookrightarrow \mathrm{Ker}(g)$ injective

if $B \twoheadrightarrow C \Longrightarrow \mathrm{Cok}(g) \twoheadrightarrow \mathrm{Cok}(h)$ surjective

theorem

Let $0 \to C'_\cdot \xrightarrow{\alpha} C_\cdot \xrightarrow{\beta} C''_\cdot \longrightarrow 0$ is a short exact sequence of algebraic complex with homomorphisms, so $\forall n \in \mathbb{N}$, we define a R module connecting homomorphism $\Delta_n : H_n(C''_\cdot) \longrightarrow H_{n-1}(C'_\cdot)$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_n(C'_\cdot) \arrow[r] \arrow[d] & Z_n(C_\cdot) \arrow[r] \arrow[d] & Z_n(C''_\cdot) \arrow[d] \arrow[dddlll, bend left] & \\
0 \arrow[r] & C'_n \arrow[r, "\alpha_n"] \arrow[d, "\partial'_n"] & C_n \arrow[r, "\beta_n"] \arrow[d, "\partial_n"] & C''_n \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "\partial'_{n-1}"] & C_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d, "\partial_{n-1}"] & C''_{n-1} \arrow[r] \arrow[d, "\partial''_{n-1}"] & 0 \\
 & C'_{n-1}/B_{n-1}(C'_\cdot) \arrow[r] \arrow[d] & C_{n-1}/B_{n-1}(C_\cdot) \arrow[r] \arrow[d] & C''_{n-1}/B_{n-1}(C''_\cdot) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Z_n''

two rows and three columns are exact, according to Snake lemma

we get the 2rd . 5th rows are also exact

so we get a commutative diagram, where the two roles are exact

and then use Snake lemma again

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \bar{\partial}'_n \arrow[r, hook, "\bar{\alpha}_n"] \arrow[d] & \operatorname{Ker} \bar{\partial}_n \arrow[r, "\bar{\beta}_n"] \arrow[d] & \boxed{\operatorname{Ker} \bar{\partial}''_n = Z_n/B_n(C''_\cdot)} \arrow[d] \arrow[ddddll, bend left, "\Delta_n"] & \\
 & C'_n/B_n(C'_\cdot) \arrow[r, hook] \arrow[d, "\bar{\partial}'_n"] & C_n/B_n(C_\cdot) \arrow[r, two heads] \arrow[d, "\bar{\partial}_n"] & C''_n/B_n(C''_\cdot) \arrow[r] \arrow[d, "\bar{\partial}''_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(C'_\cdot) \arrow[r, hook] \arrow[d] & Z_{n-1}(C_\cdot) \arrow[r, two heads] \arrow[d] & Z_{n-1}(C''_\cdot) \arrow[d] & \\
 & \boxed{Z_{n-1}/B_{n-1}(C'_\cdot) = \operatorname{Cok} \bar{\partial}'_n} \arrow[r] & \operatorname{Cok} \bar{\partial}_n \arrow[r, two heads] & \operatorname{Cok} \bar{\partial}''_n \arrow[r] & 0
\end{tikzcd}
$$

noticed that $\operatorname{Ker} \bar{\partial}_n \cong H_n(C_\cdot)$, $\operatorname{Coker} \bar{\partial}_n \cong H_{n-1}(C_\cdot)$

so we get long exact sequence

$$H_n(C'_\cdot) \xrightarrow{\bar{\alpha}_n} H_n(C_\cdot) \xrightarrow{\bar{\beta}_n} H_n(C''_\cdot) \xrightarrow{\Delta_n} H_{n-1}(C'_\cdot) \xrightarrow{\bar{\alpha}_{n-1}} H_{n-1}(C_\cdot) \xrightarrow{\bar{\beta}_{n-1}} H_{n-1}(C''_\cdot)$$

$\Delta_n : H_n(C''_\cdot) \longrightarrow H_{n-1}(C'_\cdot)$ where $H_n(C''_\cdot) = Z_n(C''_\cdot)/B_n(C''_\cdot)$

$z''_n + B_n(C''_\cdot) \longmapsto \alpha_{n-1}^{-1} \circ \partial_n \circ \beta_n^{-1}(z''_n) + B_{n-1}(C'_\cdot)$

there are two short exact sequence, and morphism φ

$$\begin{tikzcd} 0 \arrow[r] & C'_\cdot \arrow[r, "\alpha"] \arrow[d, "\varphi'"] \arrow[dr, phantom, "\circlearrowright"] & C_\cdot \arrow[r, "\beta"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowright"] & C''_\cdot \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & D'_\cdot \arrow[r, "\gamma"'] & D_\cdot \arrow[r, "\delta"'] & D''_\cdot \arrow[r] & 0 \end{tikzcd}$$

then we have

$$\begin{tikzcd} \arrow[r] & H_n(C'_\cdot) \arrow[r, "\overline{\alpha}_n"] \arrow[d, "\overline{\varphi}'_n"] \arrow[dr, phantom, "\circlearrowright"] & H_n(C_\cdot) \arrow[r, "\overline{\beta}_n"] \arrow[d, "\overline{\varphi}_n"] \arrow[dr, phantom, "\circlearrowright"] & H_n(C''_\cdot) \arrow[r, "\Delta_n"] \arrow[d, "\overline{\varphi}''_n"] \arrow[dr, phantom, "\circlearrowright"] & H_{n-1}(C'_\cdot) \arrow[r] \arrow[d, "\overline{\varphi}'_{n-1}"] & {} \\ \arrow[r] & H_n(D'_\cdot) \arrow[r, "\overline{\gamma}_n"] & H_n(D_\cdot) \arrow[r, "\overline{\delta}_n"] & H_n(D''_\cdot) \arrow[r, "\Delta_n"] & H_{n-1}(D'_\cdot) \arrow[r] & {} \end{tikzcd}$$

Čech cohomology

Let $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$, $(I, \leq)$ an index set, is an open covering of topological space, $n+1$ index $i_0\ i_1 \cdots\cdots i_n$, let $s = [i_0\ i_1 \cdots\cdots i_n]$ is regarded as a n-simplex, all this kind of simplex form a simplicial complex $K_n(I)$

so we have $C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots\cdots C_0(K) \xrightarrow{\partial_0} 0$

where $C_n(K)$ is a free Abelian group with a basis of simplex in $K_n(I)$, and $\partial_n([i_0\ i_1 \cdots\cdots i_n]) = \sum_{j=0}^{n} (-1)^j [i_0 \cdots\cdots \hat{i_j} \cdots\cdots i_n]$

for n-simplex $[i_0 i_1 \cdots i_n]$, open subset $\mathcal{U}_s = \mathcal{U}_{i_0 \cdots i_n} = \bigcap_{j=0}^{n} \mathcal{U}_{i_j}$

Let $\mathcal{F}$ is an Abelian group sheaf, let's define $C^\bullet(\mathcal{U}, \mathcal{F})$ which is a duality complex of $C_\bullet(K)$

$$C^n(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_n} \mathcal{F}(\mathcal{U}_{i_0 \cdots i_n}) = \prod_{s \in K_n(I)} \mathcal{F}(\mathcal{U}_s)$$

so $\forall f \in C^n(\mathcal{U}, \mathcal{F})$, $\{f_{i_0 \cdots i_n}\}_{i_0 < \cdots < i_n} = \{f_s\}_{s \in K_n(I)}$

$f_s = f_{i_0 \cdots i_n} \in \mathcal{F}(\mathcal{U}_{i_0 \cdots i_n})$, for $c_n = \sum_{s \in K_n(I)} m_s \cdot s \in C_n(K)$, $m_s \in \mathbb{Z}$

$f \in C^n(\mathcal{U}, \mathcal{F})$, $c_n \in C_n(K)$ the duality product

$(f, c_n) = \{m_s \cdot f_s\}_{s \in K_n(I)} \subseteq C^n(\mathcal{U}, \mathcal{F})$, with the definition of duality product, the upper boundary map $d^n : C^n(\mathcal{U}, \mathcal{F}) \to C^{n+1}(\mathcal{U}, \mathcal{F})$ such that $(d^n f, c_{n+1}) = (f, \partial_{n+1} c_{n+1})$, $f_{i_0 \cdots \hat{i_j} \cdots i_{n+1}} \in \mathcal{F}(\mathcal{U}_{i_0 \cdots \hat{i_j} \cdots i_{n+1}})$

$$(d^n f)_{i_0 \cdots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j f_{i_0 \cdots \hat{i_j} \cdots i_{n+1}} \Big|_{\mathcal{U}_{i_0 \cdots i_{n+1}}}$$

according to $\partial_n \partial_{n+1} = 0$, we have $d_{n+1} d_n = 0$, so

$$0 \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \cdots C^n(\mathcal{U}, \mathcal{F}) \xrightarrow{d^n} C^{n+1}(\mathcal{U}, \mathcal{F}) \longrightarrow$$

and we can also define the n-th Čech cohomological group

$$\check{H}^n(\mathcal{U}, \mathcal{F}) = H^n(C^\bullet(\mathcal{U}, \mathcal{F})) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$$

lemma. Let X be a topological space and $\mathcal{U}$ is a open covering $\mathcal{F}$ is an abelian group sheaf, so we have

$$\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F})$$ all global sections on X

let's proof: $\check{H}^0(\mathcal{U},\mathcal{F}) = \operatorname{Ker} d^0$, $f \in C^0$: $f = \{f_i\}_{i\in I}$

for $i<j$, $0 = (d^0 f)_{ij} = f_j - f_i$ that means for all $i,j \in \mathbb{N}$

$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so according to the 2nd axiom: Gluability

$\exists\, s \in \Gamma(X,\mathcal{F})$ such that $s|_{U_i} = f_i$ so $\operatorname{Ker} d^0 = \Gamma(X,\mathcal{F})$ □

def: sheaf holomorphism

X topological space, $\mathcal{F}$, $\mathcal{G}$ are two sheaves on X, assume $\mathcal{U} = \{U_i\}$ is the total set of all open sets on X, if there are a family of ring/group homomorphisms $\{\varphi_{U_i}\}$.

$$\varphi_{U_i} : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$$

such that the diagram commutes, $U_j \subset U_i$, we call $\varphi = \{\varphi_{U_i}\}$

$$\begin{tikzcd}
\mathcal{F}(U_i) \arrow[r, "\varphi_{U_i}"] \arrow[d, "\rho_{U_i,U_j}"'] & \mathcal{G}(U_i) \arrow[d, "\tilde{\rho}_{U_i,U_j}"'] \\
\mathcal{F}(U_j) \arrow[r, "\varphi_{U_j}"] & \mathcal{G}(U_j)
\end{tikzcd}$$

as the sheaf homomorphism from $\mathcal{F}$ to $\mathcal{G}$

def: sheaf exact sequence

Let $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ are sheaves on the topological space X

and $i_k : \mathcal{F}_k \longrightarrow \mathcal{F}_{k+1}$ is a sheaf homorphism.

$$\mathcal{F}_0 \xrightarrow{i_0} \mathcal{F}_1 \xrightarrow{i_1} \cdots\cdots \mathcal{F}_{n-1} \xrightarrow{i_{n-1}} \mathcal{F}_n \xrightarrow{i_n}$$

is called a sheaf exact sequence, if $\forall x \in X$

$$\mathcal{F}_0(x) \xrightarrow{i_0} \mathcal{F}_1(x) \xrightarrow{i_1} \mathcal{F}_2(x) \xrightarrow{i_2} \cdots\cdots$$

is a group / ring exact sequence

def: $H^n(X,\mathcal{F})$ global

Let X to be an Abelian group sheaf, we define to Čech cohomological group of $\mathcal{F}$ at global X is a direct limit

$$H^n(X,\mathcal{F}) = \varinjlim_{\mathcal{U}_i} H^n(\mathcal{U}_i,\mathcal{F})$$

$$\begin{tikzcd}
\arrow[r] & H^n(\mathcal{U}_i,\mathcal{F}) \arrow[r] \arrow[dr] & H^n(\mathcal{U}_i,\mathcal{F}) \arrow[r] \arrow[d] & H^n(\mathcal{U}_i,\mathcal{F}) \arrow[r] \arrow[dl] & {} \\
& & H^n(X,\mathcal{F}) & &
\end{tikzcd}$$

theorem

Let X be a topological space, $\mathcal{U} = \{\mathcal{U}_i\}$ is an open covering so $\forall n \geq 0$ exists homomorphism $H^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$ and the following diagram commutes

$$
\begin{tikzcd}
H^n(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & H^n(X, \mathcal{F}) \arrow[d] \\
H^n(\mathcal{U}, \mathcal{G}) \arrow[r] & H^n(X, \mathcal{G})
\end{tikzcd}
$$

introductory content : divisor

def : divisor D

on a compact Riemann surface X, a divisor can be simply defined as a formal linear combination of points on X with integer coefficients : $D = \sum n_p \cdot p$, where $p \in X$, $n_p \in \mathbb{Z}$

the degree of a divisor : $\deg(D) = \sum n_p$

if f is a meromorphic function, $(f) = \mathrm{div}(f) = \sum_p \mathrm{ord}_p(f) \cdot p$

where $\mathrm{ord}_p(f)$ represents the order of f at the point p

- p is a zero of f, $\mathrm{ord}_p(f) > 0$

  if f has a zero of order n at p, $f(z) = (z-p)^n \cdot g(z)$, where $g(z)$ is a holomorphic function at p, so $\mathrm{ord}_p(f) = n > 0$

- p is a pole of f, $\mathrm{ord}_p(f) < 0$

  if f has a pole of order n at p, $f(z) = g(z)/(z-p)^n$, $\mathrm{ord}_p(f) = -n$

- if p is neither a zero nor a pole, then $\mathrm{ord}_p(f) = 0$

def : canonical divisor K

on a compact Riemann surface X, and let ω be a non-zero meromorphic differential on X, the divisor of ω is defined as : $\mathrm{div}(\omega) = \sum_p \mathrm{ord}_p(\omega) \cdot p$ is called canonical divisor canonical divisor also represent the zeros and poles of ω

Lemma: on a compact Riemann surface, any non-constant meromorphic function has the same number of zeros and poles, when counted with multiplicity $\sum_{p} ord_p(f)$. f meromorphic on a compact Riemann surface

so if α, β are two different meromorphic 1-form, the degree of $div(\alpha)$ is equal to the degree of $div(\beta)$

def: $l(D)$

we say a divisor $D = \sum_{p} n_p \cdot p \geq 0$, if $n_p \geq 0$, $\forall p$

$$l(D) = \dim \{ \text{meromorphic function } f \text{ over } X, \ \forall p: ord_p(f) + n_p \geq 0 \}$$

the number of zeros of f $\geq$ the number of poles of D

the number of poles of f $\leq$ the number of zeros of D

$l(0) = 1$, on compact Riemann surface, holomorphic functions are constant

$l(D) = 0$, if $D < 0$

f has more zeros than poles, not exit on compact Riemann surface

$\boxed{\text{example}}$ $S^2 = \mathbb{P}^1$ dim = 1, projective space

$\mathbb{P}^1$ = the set of lines in $\mathbb{C}^2$ passing $\vec{0}$

$$\mathbb{P}^1 = \{(a,b) \neq (0,0) \text{ in } \mathbb{C}^2\} \big/ \sim \quad , \quad (a,b) \sim (\lambda a, \lambda b), \ \lambda \neq 0$$

$$= \mathbb{C} \times \mathbb{C} - \{0\} \big/ \mathbb{C}^* \ , \ \lambda \in \mathbb{C}^*$$

$$= \{[a,b] \mid (a,b) \neq \vec{0}, \ \lambda \neq 0 : [a,b] = [\lambda a, \lambda b]\}$$

on $\mathbb{P}^1$ we have two functions $F(z_0, z_1) = z_1^2$, $G(z_0, z_1) = z_0(z_0 - z_1)$

$\boxed{\text{def}}$ 1. holomorphism : complex differentiable in a neighbourhood of each point in a domain in $\mathbb{C}^n$

2. a meromorphic function on an open subset of $\mathbb{C}^n$ means a function that is holomorphic on all of the subset except for a set of isolated points

$\boxed{\text{lemma}}$ a meromorphic function can be expressed as the ratio of two holomorphic functions

F, G are holomorphic, $f = \dfrac{F}{G} = \dfrac{z_1^2}{z_0(z_0 - z_1)}$ meromorphism

f has two zeros: $2p = [z_0, 0] \longrightarrow [1, 0]$ and

f has two poles: $q_1 = [0, z_1] \longrightarrow [0, 1]$, $q_2 = [z_0, z_1] \longrightarrow [1, 1]$

symbol: $2p - q_1 - q_2$

So we construct another meomorphic function g, has the same symbol $2p - q_1 - q_2$, means 2 zero: $2p$, 2 poles: q_1, q_2

f/g: no poles and no zeros   no poles $\longrightarrow$ holomorphic

The Liouville's theorem

$f(z)$ is holomorphic over $\mathbb{C}$, and $\forall z: f(z) \not\equiv 0$

and $f(z)$ is bounded, so we have $f(z) \equiv$ const

no zeros $\longrightarrow$ constant $f/g = \lambda$

This symbol determines a meromorphic function up to a constant, we call it divisor of f, denoted (f)

Why define divisor? from number theory

if $n \mid m$, $n, m \in \mathbb{Z}$, we say m is a divisor of n

and in $\mathbb{Z}$, we have "fundamental theorem of Arithmetic"

$\forall n \in \mathbb{N}_+$, $n = \prod_i p_i^{n_i}$ a formal sum, this property is called unique factorization, so ring $\mathbb{Z}$ is a UFD

Riemann Roch theorem

Let D be a divisor on a curve X of genus g, then

$$\ell(D) - \ell(K-D) = \deg D + 1 - g$$

modern version: Let X be a curve, and $\mathcal{L}$ is a line bundle

$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$, $g = \dim H^1(X, \mathcal{O}_X)$ is genus

def Euler characteristic

Let $\mathcal{F}$ be a sheaf on a topological space X

$$\chi(\mathcal{F}) = \sum_{i=0}^{\dim X} (-1)^i \cdot \dim H^i(X, \mathcal{F})$$

Let X be a smooth, projective curve over $\mathbb{C}$, $\dim = 1$

$\chi(\mathcal{F}) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F})$

1. first step check "trivial line bundle" $\mathcal{L} = \mathcal{O}_X$

(1). $H^0(X, \mathcal{O}_X)$, the set of global regular function, we assume the curve over field $k = \mathbb{C}$, so the global regular function on algebraic closed $\mathbb{C}$ is only constant function, so $\dim H^0(X, \mathcal{O}_X) = 1$

according to Liouville theorem

(2) $H^1(X, \mathcal{O}_X) = g$ the definition of genus

(3) $\deg \mathcal{O}_X = 0$, regular (holomorphic) $\longrightarrow$ no zeros, poles

def: line bundle degree

the degree of line bundle $\mathcal{L}$ on a curve over field k is defined as the degree of the divisors of any nonzero rational section s of $\mathcal{L}$

from trivial line bundle we get trivial section as constant, which has no zero, pole, so $\deg(\mathcal{L}) = 0$

2. second step, use induction with short exact sequence

$\forall p \in X$, consider $\mathcal{L}(p) = \mathcal{L} \otimes \mathcal{O}_X(p)$ and skyscraper sheaf k_p

def: skyscraper sheaf

Let X be a topological space and $x \in X$, skyscraper sheaf $\mathcal{F}_x$ open set $\mathcal{U} \subset X$, $\mathcal{A}$ is an algebraic structure, here is $\mathbb{C}$

$$\text{define } \mathcal{F}_x(\mathcal{U}) = \begin{cases} \mathcal{A}, & \text{if } x \in \mathcal{U} \\ 0, & \text{if } x \notin \mathcal{U} \end{cases}$$

if $x \in V \subset \mathcal{U}$, $res_{\mathcal{U}V} = id_{\mathcal{A}}$, if $x \notin V$, $res = 0$ map

explaination for k_p: $H^0(X, k_p) = k$, $\forall i > 0$, $H^i(X, k_p) = 0$

X is a topological space, a point $p \in X$, k is a residue field at p, the stalk of Skyscraper sheaf k_p at p is k

$$k_p(\mathcal{U}) = \begin{cases} k, & \text{if } p \in \mathcal{U} \\ 0, & \text{if } p \notin \mathcal{U} \end{cases}$$

so $H^0(X, k_p) = k$, $H^1(X, k_p) = 0$

$\mathcal{L}_p = \mathcal{L} \otimes \mathcal{O}_X(p)$, line bundle $\mathcal{O}_X(p)$ corresponding divisor $D = p$

global section is meromorphic functions f satisfying $\mathrm{div}(f) + p \geq 0$

function can have a simple pole only at p

consider a short sequence $0 \longrightarrow \mathcal{L} \xrightarrow{\alpha} \mathcal{L}_p \xrightarrow{\beta} k_p \longrightarrow 0$

well it is an exact sequence : $\mathrm{Im}(\alpha) = \alpha(\mathcal{L}) = \mathrm{Ker}(\beta)$

means $\mathcal{L} \overset{\alpha}{\hookrightarrow} \mathcal{L}_p$ injective, $\mathcal{L}_p \xrightarrow{\beta} k_p$ surjective

- $\mathcal{L}$ is a subsheaf of $\mathcal{L}_p$
- k_p is a qutient sheaf of $\mathcal{L}_p$: $k_p \sim \mathcal{L}_p / \mathcal{L}$
- $\mathrm{Ker}\, \beta = \mathrm{Im}\, \alpha$

according to Čech cohomology theorem we have $C^\bullet(X, \mathcal{F})$

$0 \to \mathcal{L} \to \mathcal{L}_p \to k_p \to 0$ a short exact sequence, we have a long exact sequence ; $\exists\, \delta^n : H^n(X, k_p) \longrightarrow H^{n+1}(X, \mathcal{L})$

$$0 \longrightarrow H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}_p) \longrightarrow H^0(X, k_p) \xrightarrow{\delta^0} H^1(X, \mathcal{L})$$

rows are exact

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^{n-1}(X,\mathcal{L}) \arrow[r, hook, "\alpha^{n-1}"] \arrow[d] & C^{n-1}(X,\mathcal{L}_p) \arrow[r, "\beta^{n-1}"] \arrow[d] & C^{n-1}(X,k_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{n}(X,\mathcal{L}) \arrow[r, hook, "\alpha^{n}"] \arrow[d, "\partial"] & C^{n}(X,\mathcal{L}_p) \arrow[r, "\beta^{n}"] \arrow[d, "\partial"] & C^{n}(X,k_p) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^{n+1}(X,\mathcal{L}) \arrow[r, hook, "\alpha^{n+1}"] \arrow[d, "\partial"] & C^{n+1}(X,\mathcal{L}_p) \arrow[r, "\beta^{n+1}"] \arrow[d, "\partial"] & C^{n+1}(X,k_p) \arrow[r] \arrow[d] & 0 \\
 & \, & \, & \, &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & g \in \arrow[rr] & & h \in Z^n(X,k_p) \arrow[d] \\
Z^{n+1}(X,\mathcal{L}) \arrow[r, leftrightarrow] & f \in \arrow[r] \arrow[d] & \partial g \in \arrow[u] \arrow[rr] \arrow[d] & & 0 \in \\
 & \partial f \arrow[r] & 0 & &
\end{tikzcd}
$$

$\partial f = 0$

$f \in C^{n+1}(X,\mathcal{L})$: $\exists\, \partial g \in C^{n+1}(X,\mathcal{L}_p)$ such that $\partial g = \alpha^{n+1}(f)$

where $g \in C^n(X,\mathcal{L}_p)$.

$$0 = \partial(\partial g) = \partial(\alpha f) = \alpha(\partial f) \text{ , } \alpha \text{ is injective so } \partial f = 0$$

we know short exact sequence $\operatorname{Im}\alpha = \operatorname{Ker}\beta$: $\beta^{n+1}(\alpha^{n+1} f) = 0$

so from Snake lemma and its corollary

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{L}) \arrow[r] & H^0(X,\mathcal{L}_p) \arrow[r] & H^0(X,k_p) \arrow[dlll, "\delta^0"'] & \\
H^1(X,\mathcal{L}) \arrow[r] & H^1(X,\mathcal{L}_p) \arrow[r] & H^1(X,k_p) \arrow[r] & 0 &
\end{tikzcd}
$$

calculate Euler characteristic (on a projective curve)

$\chi(k_p)$: $H^0(X, k_p) = k$, and $\dim H^0(X, k_p) = 1$

$\dim H^1(X, k_p) = 0$ so $\chi(k_p) = \dim H^0(k_p) - \dim H^1(k_p) = 0$

from $0 \to \mathcal{L} \to \mathcal{L}_p \to k_p \to 0$ so $\chi(\mathcal{L}_p) = \chi(\mathcal{L}) + \chi(k_p)$

$\chi(\mathcal{L}_p) = \chi(\mathcal{L}) + \chi(k_p) = \chi(\mathcal{L}) + 1$,

$\deg(\mathcal{L}_p) = \deg \mathcal{L} + 1$

divisor $\overset{1}{\underset{2}{\rightleftharpoons}}$ line bundle

1. divisor $D = \sum_i n_i p_i$, $\mathcal{O}_X(D)$. $\forall f \in \mathcal{O}_X(D)$ is meromorphic functions satisfying $\operatorname{div}(f) + D \geq 0$

2. from a line bundle $\mathcal{L}$ we can form a divisor : $\mathcal{L} \cong \mathcal{O}_X(D)$

if $\mathcal{L} = \mathcal{O}_X(D)$. so $\mathcal{L}_p = \mathcal{L} \otimes \mathcal{O}_X(p)$

corresponding divisor $D + p$. means $\mathcal{L}_p \cong \mathcal{O}_X(D+p)$

$\deg(D) = \sum_i n_i$, $\mathcal{L} = \mathcal{O}_X(D)$: $\deg \mathcal{L} = \deg D$

$\deg(\mathcal{L}_p) = \deg(D + p) = \deg D + \deg p = \deg \mathcal{L} + 1$

because p is a point

suppose Riemann - Roch theorem is right for $\mathcal{L}$

$$\chi(\mathcal{L}_p) = \chi(\mathcal{L}) + 1 \xrightarrow{RR} (\deg \mathcal{L} + 1 - g) + 1$$
$$= \deg(\mathcal{L}_p) + 1 - g$$

third step: Serre duality theorem

theorem: Serre duality theorem   local $\Longleftrightarrow$ global (duality)

let X be a smooth, $\dim = n$, projective, algebraic variety

here X is a smooth projective curve, $\mathcal{L}$ is line bundle on X

ω_X is a canonical sheaf over X, so we have

$$H^0(X,\mathcal{L}) = H^1(X,\mathcal{L}^\vee \otimes \omega_X)^\vee$$

$$H^1(X,\mathcal{L}) = H^0(X,\mathcal{L}^\vee \otimes \omega_X)^\vee$$

where $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L},\mathcal{O}_X)$ duality sheaf of $\mathcal{L}$, $\mathcal{L}^\vee = \mathcal{L}^{-1}$

$(\cdots)^\vee$ means duality vector space

def: canonical sheaf ω_X on a curve X

$\omega_X = \Omega_X$, where Ω_X is the cotangent bundle on X

point $p \in X$, the fiber of cotangent bundle at p is the cotangent space T_p^*X, which consists of all the linear functions acting on T_pX

$\forall$ section $g \in \omega_X$, $g = f(z)\,dz$ (local coordinate), where f is a regular function

back to the proof of Riemann-Roch theorem

$$H^1(X,\mathcal{L}) = H^0(X,\mathcal{L}^{-1} \otimes \omega_X)^\vee$$

put into Riemann Roch $\chi(\mathcal{L}) = H^0(X,\mathcal{L}) - H^1(X,\mathcal{L})$

$$\chi(\mathcal{L}) = H^0(X, \mathcal{L}) - H^0(X, \mathcal{L}^{-1} \otimes \omega_X)$$

$$= \deg \mathcal{L} + 1 - g$$

sheaf $\mathcal{L} \longrightarrow$ divisor D

sheaf $\mathcal{L}^{-1} \longrightarrow$ divisor $-D$ $\begin{cases} \text{zero} \longrightarrow \text{pole} \\ \text{pole} \longrightarrow \text{zero} \end{cases}$

sheaf $\omega_X \longrightarrow$ divisor K

sheaf $\omega_X \otimes \mathcal{L}^{-1} \longrightarrow$ divisor $K - D$

Riemann Roch theorem

Let D be a divisor on a curve X of genus g, then

$$\ell(D) - \ell(K-D) = \deg D + 1 - g$$

$$\ell(D) = \dim \left\{ \text{meromorphic function } f \text{ over } X, \ \forall p : \ \mathrm{ord}_p(f) + n_p \geq 0 \right\}$$

$$= \dim H^0(X, \mathcal{L})$$

apply $\mathcal{L} \longrightarrow \mathcal{O}_X$

MATH 5261: Algebraic Geometry II. 2025 Spring

Lecture 1: Overview & Introduction to Homological Algebra. jianggy@ust.hk

GOAL of this semester: use cohomological tools to study algebraic geometry.

$\sim$ roughly Hartshorne Chapter III (+ ~~rth~~ II, ...)

main themes: I. Serre duality. II. Flat, smooth morphisms.

Riemann-Roch, ... Family of schemes, Base-change, ...

tools to develop: 0. Cohomology theory (of sheaves)

§1. Riemann-Roch

Let X be a smooth projective curve over $k = \mathbb{C}$.

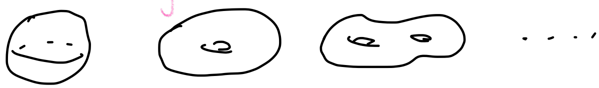
$X \subseteq \mathbb{P}^n_{\mathbb{C}}$ for some n $\quad$ $\dim X = 1$

$\dim T_x X = 1 \ \forall x \in X$ closed.

$\xleftrightarrow{iso}$ $X(\mathbb{C})$: Riemann surfaces. $\dim_{\mathbb{R}}(X(\mathbb{C})) = 2$ Closed, oriented real 2 dim'l manifold.

compact

on a compact connected Riemann surface, all the holomorphic functions are constant function, dim = 1



$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$

Def'n: • A divisor on X is a formal sum

$$D = \sum_{i=1}^{m} n_i x_i, \quad n_i \in \mathbb{Z},\ x_i \in X \text{ closed points.}$$

$$= \sum_{\text{finite}} n_x \cdot x, \quad n_x \in \mathbb{Z},\ x \in X \text{ closed.}$$

$$\deg(D) := \sum_{i=1} n_i \in \mathbb{Z} \quad \text{degree of } D.$$

• For a divisor $D = \sum_x n_x \cdot x$, define a $\mathcal{O}_X$-module $\mathcal{O}_X(D)$:

$$\ell(D)\ \Gamma(U, \mathcal{O}_X(D)) = \{ f \in K(X) \mid \forall x \in U \text{ closed}, \underbrace{\mathrm{ord}_x(f)}_{\text{order at } x} + \underbrace{n_x}_{} \geq 0 \}$$

$K(X)$: function field of X "meromorphic fns"

$\mathrm{div}(f)|_U + D|_U \geq 0.$

Fact: $\mathcal{O}_X(D)$ is a line bundle. (locally free of rank 1).

E.g.: • $D=0$, $\mathcal{O}_X(0)=\mathcal{O}_X$.

• $D=-x$, $\Gamma(U,\mathcal{O}_X(D))=\{f\in\mathcal{O}_X(U) \mid f_x\in m_x\subseteq\mathcal{O}_{X,x}\}$. $\mathcal{O}_X(D)\subseteq\mathcal{O}_X$.

regular f'ns vanishing at x

$\mathcal{O}_X(D)=\mathcal{I}_x$ ideal sheaf of x.

• $D=x$, $\mathcal{O}_X\subseteq\underbrace{\mathcal{O}_X(D)}\subseteq K(X)$

subsheaf of meromorphic f'ns allowing poles of order 1 only at $x\in X$.

Generally: $D=a_1x_1+\cdots+a_nx_n-b_1y_1-\cdots-b_my_m$, $a_i,b_j>0$

$$\Gamma(X,\mathcal{O}_X(D))=\left\{f\in K(X)\;\middle|\;\begin{array}{l}f \text{ allow poles of order} \le a_i \text{ at } x_i\\ f \text{ has vanishing order} \ge b_j \text{ at } y_j\end{array}\right\}$$

THM (Riemann-Roch): $\forall$ divisor D on a smooth projective curve $X/\mathbb{C}$

$$\underbrace{\dim\Gamma(X,\mathcal{O}_X(D))}-\dim\underbrace{\Gamma(X,\mathcal{O}_X(K-D))}=\underbrace{1-g+\deg(D)}.$$

algebraic / holomorphic / analytic inform   topological inform

Here • K canonical divisor: $\mathcal{O}_X(K)=\Omega^1_X$ cotangent bundle

Ω^1_X is a sheaf of $\mathcal{O}_X$-modules s.t. $\Omega^1_X\otimes k(x)\cong m_x/m_x^2$

Kähler differentials (later).

• $g=\dim_{\mathbb{C}}\Gamma(X,\Omega^1_X)=\dim_{\mathbb{C}}\Gamma(X,\mathcal{O}_X(K))$. genus.

$g=\#$ of holes of $X(\mathbb{C})=\frac{1}{2}\dim H^1(X(\mathbb{C}))$

The most important result in studying curves. Many proofs.

Modern simple proof in algebraic geometry:

"cohomology groups"

1) Machinery of sheaf cohomology: $H^i(X,\mathcal{F})$, $\mathcal{F}$ (coherent) sheaf

$$\Gamma(X,\mathcal{F})=H^0(X,\mathcal{F}).$$

For any short exact sequence of (coherent) sheaves

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0.$$

It produces a long exact sequence

$$0 \to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to H^0(X,\mathcal{F}_3) \to H^1(X,\mathcal{F}_1) \to H^1(X,\mathcal{F}_2) \to H^1(X,\mathcal{F}_3) \to \cdots$$

X curve $\uparrow$ 0

- Vanishing theorems: $H^i(X,\mathcal{F}) = 0$ for $i \gg 0$.

  In fact, X curve, $H^i(X,\mathcal{F}) = 0$ for $i \neq 0, 1$.

- Finiteness results: X projective $/\mathbb{C}$ (in fact, proper $/\mathbb{C}$ is ✓)

  $\Rightarrow \dim_{\mathbb{C}} H^i(X,\mathcal{F}) < \infty$.

Euler $\Rightarrow$ $\chi(X,\mathcal{F}_1) - \chi(X,\mathcal{F}_2) + \chi(X,\mathcal{F}_3) = 0$.

$$\chi(X,\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{F})$$

$$= \dim_{\mathbb{C}} \underbrace{H^0(X,\mathcal{F})}_{= \Gamma(X,\mathcal{F})} - \dim_{\mathbb{C}} H^1(X,\mathcal{F}), \quad X \text{ curve}.$$

Apply to the short exact sequence (say $D \geq 0$).

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0. \qquad X \text{ curve}$$

$$\chi(X,\mathcal{O}_X(D)) = \chi(X,\mathcal{O}_X) + \deg(D).$$

2). Serre duality:

X curve. $H^1(X,\mathcal{F}) \cong H^0(X,\mathcal{F}^\vee \otimes \Omega_X^1)^\vee$ (dual vector space $/\mathbb{C}$), $\mathcal{F}$ vector bundle

$$\Rightarrow \begin{cases} H^1(X,\mathcal{O}_X(D)) \cong H^0(X,\mathcal{O}_X(K-D))^\vee. \\ H^1(X,\mathcal{O}_X) \cong \Gamma(X,\Omega_X^1)^\vee. \end{cases}$$

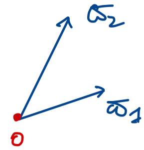
Now: $\chi(X,\mathcal{O}_X) = \dim \underbrace{\Gamma(X,\mathcal{O}_X)}_{1} - \dim \underbrace{\Gamma(X,\Omega_X^1)}_{g}$. "$\square$"

In general: X ~~smooth~~ projective $/k$ of dim n, $\mathcal{F}$ ~~v.b.~~ coherent

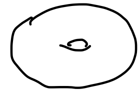
Serre's duality: $H^i(X,\mathcal{F}) \cong H^{n-i}(X,\mathcal{F}^\vee \otimes \underbrace{\Lambda^n \Omega_X^1}_{\omega_X})^\vee$. $\leftarrow$ will prove this.

$H^i(X,\mathcal{F}) \cong \mathrm{Ext}^{n-i}(\mathcal{F},\omega_X)^\vee$.

Example: X = elliptic curve over $k = \mathbb{C}$.



Know: $X \overset{\text{analytically}}{=\!=} \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$.



$\mathcal{L} := \mathcal{O}(0)$. $\mathcal{L}^{\otimes d} = \mathcal{O}(d.0)$, $d>0$.

$\deg((\mathcal{L}^{\otimes d})^\vee \otimes \Omega_X^1) = -d < 0$ if $d>0$.

[Ex: $\Rightarrow$ $\Gamma((\mathcal{L}^{\otimes d})^\vee \otimes \Omega_X^1) = 0$.]

Then R.R. $\Rightarrow$ $\dim \Gamma(X, \mathcal{L}^{\otimes d}) = d$, $d>0$.

d	$\dim \Gamma(X, \mathcal{L}^d)$	Basis	
1	1	1	
2	2	$\wp, 1$	$\rightsquigarrow$ $X \ni z \mapsto [\wp(z), 1]$, $X \longrightarrow \mathbb{P}^1$?
3	3	$\wp, \wp', 1$	$\rightsquigarrow$ $X \ni z \longmapsto [\wp(z), \wp'(z), 1]$, $X \longrightarrow \mathbb{P}^2$? $(\star)$
4	4	$\wp, \wp^2, \wp', 1$	Question: $\wp'' \in \Gamma(4\cdot 0)$? lin. rel'n ? $(**)$
...	...	......	

$(\star)$: Recall: $X \to \mathbb{P}^n$ $\leftrightsquigarrow$ $\mathcal{L} \to X$ line bundle & $(n+1)$ sections $\mathcal{O}_X^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$ s.t. gen. $\mathcal{L}$.

- For $(\star)$: Image is "homogenize"

$X \simeq \{ y^2 = 4x^3 - g_2 x - g_3 \} \subseteq \mathbb{P}^3$. $y = \wp'(z)$, $x = \wp(z)$. "Weierstrass form".

- $$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

$\begin{cases} \text{• double periodic mero. fn.} \\ \text{• has a pole of degree 2 at } 0 \in X. \end{cases}$ $(\Leftrightarrow$ $\wp(z) \in \Gamma(X, \mathcal{O}(2\cdot 0))$).

- $$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}.$$ $\wp'(z) \in \Gamma(X, \mathcal{O}(3\cdot 0))$.

- $$g_2 = 60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^2}, \quad g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^3}.$$ $\square$

$(**)$: prove that $\wp'' = 6\wp^2 - \frac{g_2}{2}$.

§2. Prelude to derived functors : Tor.

Riemann-Roch: want to know $\Gamma(X, \mathcal{F})$: this information alone : hard.

once corrected by missing hidden information about $\mathcal{F}$:

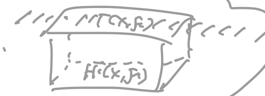
$$H^i(X, \mathcal{F})$$

Then get nice & complete understanding.

$H^i(X, \mathcal{F})$: derived functor of $\Gamma(X, \mathcal{F})$.

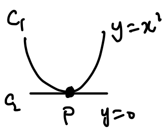
"hidden information that is determined by $\Gamma(X, \mathcal{F})$"

They are already there.
But we need to do some extra works to exact / derive these inform.

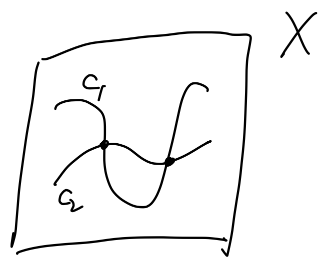


Another Example: Serre's intersection formula*

E.g.



$X = \mathbb{C}^2$



intersection multiplicity

$$\mathrm{mult}_p(C_1, C_2) = 2 = \dim_k \left(\mathcal{O}_{C_1,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{C_2,p} \right)$$

$$k[x,y]/(y) \otimes k[x,y]/(y-x^2) \simeq k[x]/(x^2).$$

Is that always true $C_1, C_2 \subseteq X$ – smooth, $C_1 \cap C_2 = \{p_1, p_2, \dots, p_n\}$
(& $\dim C_1 + \dim C_2 = X$)

$$\mathrm{mult}_p(C_1, C_2) \overset{?}{=} \dim_k \left(\mathcal{O}_{C_1,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{C_2,p} \right) ?$$

No for higher dim'l

Serre: $\mathrm{mult}_p(C_1, C_2) = \sum_i (-1)^i \dim_k \mathrm{Tor}_i^{\mathcal{O}_{X,p}} (\mathcal{O}_{C_1,p}, \mathcal{O}_{C_2,p})$.

(positivity remains open).
≥ 0 : Gabber 1995.

$\mathrm{Tor}_i^R(M, N)$: derived functor of $M \otimes_R N$.

Tor : "derived functor of $\otimes$"

R: (commutative) ring. N: R-module.

Then: $\_\otimes_R N$ is not exact, but only right exact:

- If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then for any R-module N:

upshot:
$$\cdots \xrightarrow{\delta} \mathrm{Tor}_2(M'',N) \xrightarrow{\delta} \mathrm{Tor}_1(M',N) \to \mathrm{Tor}_1(M,N) \to \mathrm{Tor}_1(M'',N) \xrightarrow{\delta} M'\otimes_R N \to M\otimes_R N \to M''\otimes_R N \to 0$$
exact

this might NOT be injective.

Rmk (Flatness). N is flat $\overset{\text{def'n}}{\iff}$ $\_\otimes_R N$ is exact

i.e. $\forall\ 0\to M' \to M \to M'' \to 0$ short exact seq (s.e.s.)

$\Rightarrow 0 \to M'\otimes_R N \to M\otimes_R N \to M''\otimes_R N \to 0$ short exact seq. $\square$

Def'n: • A free R-module $R^{\oplus S} = \bigoplus_{s\in S} R\cdot s$ S basis.

- A free resolution of an R-module M is a complex of free modules $F_* = (F_n, d)_{n\geq 0}$ together with a R-module map $\varepsilon: F_0 \to M$ s.t.
$$\cdots F_3 \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} M \to 0$$
is exact.

- Define $\mathrm{Tor}_i^R(M,N) = H_i(F_* \otimes_R N,\ \underset{\substack{\| \\ \partial_n}}{d_n \otimes \mathrm{id}_N})$

$= \text{Homology of } \left(\cdots \to F_2\otimes_R N \xrightarrow{\partial_2} F_1\otimes_R N \xrightarrow{\partial_1} F_0\otimes_R N\right)$ at i^{th} stage.

$= \dfrac{\mathrm{Ker}(\partial_i)}{\mathrm{Im}(\partial_{i+1})}$. $\quad\cdots\quad 2 \quad 1 \quad 0$ homological degree $\square$

Exercise: $\mathrm{Tor}_0^R(M,N) = M\otimes_R N$.

Lemma 1.1 (Free resolutions exist). Let M be a R-module. Then M admits a free resolution F_*.

Proof: Take
$$F_0 = \bigoplus_{m\in M} R\cdot m \xrightarrow{\ \varepsilon\ } M .$$
$$rm \longmapsto r\cdot m$$

(If M is finitely generated, $M = \sum_{i=1}^{n} r\cdot e_i$, then we can take $F_0 = \bigoplus_{i=1}^{n} R e_i$ to be a finite free module)

Set $M_0 = \mathrm{Ker}(\varepsilon)$. Then take free module $F_1 \overset{\varepsilon_1}{\twoheadrightarrow} M_0$.

Inductively, given a module M_{n-1}, take free module F_n with
$$F_n \overset{\varepsilon_n}{\twoheadrightarrow} M_{n-1}.$$

Let d_n = composition $(F_n \twoheadrightarrow M_{n-1} \hookrightarrow F_{n-1})$.

Then $F_* = (F_n, d_n)$ with $F_0 \overset{\varepsilon}{\twoheadrightarrow} M$ is a free resol'n of M.

$$\begin{tikzcd}
 & & & M_2 = \mathrm{Ker}(\varepsilon_1) \arrow[dr, hook] & & & & \\
\cdots & F_3 \arrow[r] \arrow[dr, two heads] & F_2 \arrow[rr] \arrow[ur, two heads, "\varepsilon_2"] & & F_1 \arrow[rr] \arrow[dr, two heads, "\varepsilon_1"] & & F_0 \arrow[r, two heads, "\varepsilon"] & M \\
 & & M_3 = \mathrm{Ker}(\varepsilon_2) \arrow[u, hook] & & & M_1 = \mathrm{Ker}(\varepsilon) \arrow[ur, hook] & &
\end{tikzcd}$$

$\square$

Lemma 1.2 (Free resolutions are "unique up to homotopy")

Let $F_* \xrightarrow{\varepsilon} M$, $G_* \xrightarrow{\eta} N$ be 2 free resol'ns.

Given any map $f: M \to N$.

(1) There exists a chain map $f_*: F_* \to G_*$ lifting $f: M \to N$.

i.e. $\eta \circ f_0 = f \circ \varepsilon$.

(2) Any two chain maps $f_*, f_*': F_* \to G_*$ lifting $f: M \to N$ are chain homotopy equivalence:

$\exists\, h_i: F_i \to G_{i+1}$

s.t. $f_i - f_i' = d_{i+1}' h_i + h_{i-1} d_i$.

$(f_* - f_*' = dh + hd)$

$$\begin{tikzcd}
\cdots & F_2 \arrow[r] \arrow[d, shift right, "\exists"'] \arrow[d, shift left] & F_1 \arrow[r] \arrow[d, shift right, "\exists"'] \arrow[d, shift left] \arrow[dl, "h_1"'] & F_0 \arrow[r] \arrow[d, shift right, "\exists"'] \arrow[d, shift left] \arrow[dl, "h_0"'] & M \arrow[r] \arrow[d, "f"] \arrow[dl, "h_{-1}"'] & 0 \\
\cdots & G_2 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Proof: (1).

$$\begin{tikzcd} F_0 \arrow[d, dashed, "\exists ?"'] \arrow[dr, "f\circ\varepsilon"] & \\ G_0 \arrow[r, "\eta"'] & N \end{tikzcd} \qquad \begin{tikzcd} F_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, dashed, "f_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ G_0 \arrow[r, two heads, "\eta"] & N \arrow[r] & 0 \end{tikzcd}$$

(*) Free modules are "projective": for any surjective map $B \overset{\alpha}{\twoheadrightarrow} C$ &
any $\gamma : F \to C$, F free $\exists \beta$ s.t. $\alpha\circ\beta = \gamma$.

$$\begin{tikzcd} F \text{ - free} \arrow[d, dashed, "\exists\beta"'] \arrow[dr, "\forall\gamma"] & \\ B \arrow[r, two heads, "\forall\alpha"'] & C \end{tikzcd}$$

Proof of (*): $F = \bigoplus_i Re_i$ has basis $\{e_i\}$.

For each i, consider $\gamma(e_i) \in C$. $\exists$ $b_i \in B$, $\alpha(b_i) = \gamma(e_i)$.

Define β: $e_i \mapsto b_i$ $\square$

Hence $\exists$ $f_0 : F_0 \to G_0$ s.t. $\eta\circ f_0 = f\circ\varepsilon$.

Next step:

$$\begin{tikzcd} F_1 \arrow[d, dashed, "\exists f_2"'] \arrow[dr, "f_1\circ d_1"] & \\ G_1 \arrow[r, two heads, "d_1'"'] & \mathrm{Ker}(\eta) \end{tikzcd} \qquad \begin{tikzcd} F_1 \arrow[r, "d_1"] \arrow[d, dashed, "f_2"'] & F_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, "f_1"] & M \arrow[r] \arrow[d, "f"] & 0 \\ G_1 \arrow[r, "d_1'"] & G_0 \arrow[r, two heads, "\eta"] & N \arrow[r] & 0 \end{tikzcd}$$

Inductively: $\exists$ $f_n : F_n \to G_n$ s.t. $d'\circ f_n = f_{n-1}\circ d$. $\square$ of (1).

Notice: we only used $F_\cdot$ is a chain complex of free modules &
$G_\cdot \xrightarrow{\eta} N$ is any resol'n.

(2). $\eta(f_0 - f_0') = f\circ\varepsilon - f\circ\varepsilon = 0$

$$\Rightarrow \quad \begin{tikzcd} & F_0 \arrow[dl, dashed, "\exists h_0"'] \arrow[d, "f_0 - f_0'"] \\ G_1 \arrow[r, two heads, "d'"'] & \mathrm{Ker}(\eta) \end{tikzcd} \qquad \begin{tikzcd} & F_0 \arrow[r, two heads, "\varepsilon"] \arrow[dl, dashed, "h"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_0'"] & M \arrow[d, "f"] \\ G_1 \arrow[r, "d'"'] & G_0 \arrow[r, two heads, "\eta"'] & N \end{tikzcd}$$

Inductively: $n \geq 1$ (set $F_{-1} = M$, $h_{-1} = 0$, $G_{-1} = N$):

Suppose $f_{n-1} - f'_{n-1} = d'h + hd$

Want: $f_n - f_n' = d'h + hd$.

$$\begin{tikzcd}
 & F_n \arrow[r, "d"] \arrow[d, "f_n - f_n'"] \arrow[dl, dashed] & F_{n-1} \arrow[r, "d"] \arrow[d, "f_{n-1} - f'_{n-1}"] \arrow[dl, "h_{n-1}"] & F_{n-2} \arrow[dl, "h"] \\
G_{n+1} \arrow[r, "d'"'] & G_n \arrow[r, "d'"'] & G_{n-1} &
\end{tikzcd}$$

Compute: $d'(f_n - f_n' - h_{n-1} d)$

$= (f_{n-1} - f'_{n-1}) d - d' h_{n-1} d$

$= (f_{n-1} - f'_{n-1} - d' h_{n-1}) \cdot d$

$= h_{n-2} \circ d \circ d = 0$.

Hence
$$\begin{tikzcd}
 & F_n \arrow[d, "f_n - f_n' - hd"] \arrow[dl, dashed, "\exists h_n"'] \\
G_{n+1} \arrow[r, two heads] & \operatorname{Ker}(d'_{n-1})
\end{tikzcd}$$

$\square$

Rmk: Only use $F_\bullet \to M$ is a <u>projective</u> chain complex & $G_\bullet \to N$ any resol'n. Here: P is a <u>projective</u> module if $(*)$ holds, i.e.

$$\begin{tikzcd}
P \arrow[d, dashed, "\exists \beta"'] \arrow[dr, "\forall \gamma"] & \\
B \arrow[r, two heads, "\forall \alpha"'] & C
\end{tikzcd}$$

$\forall \alpha : B \twoheadrightarrow C$, $\forall \gamma : P \to C$

$\exists \beta : P \to B$ s.t. $\gamma = \alpha \circ \beta$.

$\square$

Consequences:

Cor 1.3: ① For any $f: M \to M'$ R-module map. Take free resol'ns $F_* \to M$ & $F_*' \to M'$ & chain map $f_*: F_* \to F_*'$ lifting f.

Then $\forall N$, $H_i(F_* \otimes N) \xrightarrow{H_i(f_* \otimes id_N)} H_i(F_*' \otimes N)$ depends only on f & indep. of lifting f_*.

② Take any two free resolutions F_*, F_*' of M. For any N, $\exists$ can. isom. $H_i(F_* \otimes N) \xrightarrow{\sim} H_i(F_*' \otimes N)$.

$$\begin{tikzcd}
F_* \arrow[r] \arrow[d, "\exists"'] & M \arrow[d, "id"] \\
F_*' \arrow[r] \arrow[d, "\exists"'] & M \arrow[d, "id"] \\
F_* \arrow[r] & M
\end{tikzcd}$$

Combine ① & ②: $\operatorname{Tor}_i^R(M,N)$ is independent of the free resol'n $F_* \to M$ & $\operatorname{Tor}_i^R(-,N)$ is functorial:

$\operatorname{Tor}_i^R(-,N) : \operatorname{Mod}_R \longrightarrow \operatorname{Mod}_R$.

$\square$

Prop 1.4 (Short exact sequences $\Rightarrow$ Long exact sequences)

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then $\forall$ R-mod N, $\exists$ long exact sequence of R-modules:

$$\cdots \cdots \to \mathrm{Tor}_2(M',N) \to \mathrm{Tor}_2(M,N) \to \mathrm{Tor}_2(M'',N) \to \mathrm{Tor}_1(M',N) \to \mathrm{Tor}_1(M,N) \to \mathrm{Tor}_1(M'',N) \xrightarrow{\delta} M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

To prove Prop. 1.4, we need several ingredients:

Weibel:

> The key tool in constructing the connecting homomorphism is our next result, the Snake Lemma. We will not print the proof in these notes, because it is best done visually. In fact, a clear proof is given by Jill Clayburgh at the beginning of the movie Its My Turn (Rastar-Martin Elfand Studios, 1980). As an exercise in "diagram chasing" of elements, the student should find a proof (but privately—keep the proof to yourself!).

Lemma 1.5 (Snake lemma): Given commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ker}(f) \arrow[r] \arrow[d] & \mathrm{Ker}(g) \arrow[r] \arrow[d] & \mathrm{Ker}(h) \arrow[d] & \\
 & A' \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & B' \arrow[r, "p"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \mathrm{Cok}(f) \arrow[r] & \mathrm{Cok}(g) \arrow[r] & \mathrm{Cok}(h) &
\end{tikzcd}$$

if rows are exact.

Then $\exists$ exact sequence $\mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{Cok}(f) \to \mathrm{Cok}(g) \to \mathrm{Cok}(h)$.

Here: $\partial(c') = i^{-1} g\, p^{-1}(c')$, $c' \in \mathrm{Ker}(h)$.

Moreover, if $A' \hookrightarrow B' \Rightarrow \mathrm{Ker}(f) \hookrightarrow \mathrm{Ker}(g)$.
if $B \twoheadrightarrow C \Rightarrow \mathrm{Cok}(g) \twoheadrightarrow \mathrm{Cok}(h)$.

Exercise: find your own proof of the snake's lemma. $\square$

Thm 1.6 (Long exact sequence of cohomology).

Let $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ be a short exact sequence of chain complexes. Then there exists a long exact sequence

$$\cdots \xrightarrow{f} H_{n+1}(B) \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

∂: connecting homomorphisms.

Proof: $Z_n(A) := \mathrm{Ker}(A_n \xrightarrow{d} A_{n-1}) \subseteq A_n$ n-cycles.

$B_n(A) := \mathrm{Im}(A_{n+1} \xrightarrow{d} A_n) \subseteq A_n$ n-boundaries

$$B_n(A) \subseteq Z_n(A) \subseteq A_n.$$

$H_n(A) = Z_n(A)/B_n(A)$ nth homology.

First, consider

$$\begin{array}{ccccccccc} 0 \to & A_n & \to & B_n & \to & C_n & \to 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} & \to 0 \end{array}$$

Snake's lemma $\Rightarrow$

$$0 \to Z_n(A) \to Z_n(B) \to Z_n(C)$$
$$\to A_{n-1}/dA_n \to B_{n-1}/dB_n \to C_{n-1}/dC_n \to 0.$$

exact

$\Downarrow$

Secondly, consider

$$\begin{array}{ccccccc} & \frac{A_n}{dA_{n+1}} & \to & \frac{B_n}{dB_{n+1}} & \to & \frac{C_n}{dC_{n+1}} & \to 0 \\ & d\downarrow & & d\downarrow & & d\downarrow & \\ 0 \to & Z_{n-1}(A) & \to & Z_{n-1}(B) & \to & Z_{n-1}(C) & \end{array}$$

rows exact

Snake's lemma (x2) $\Rightarrow$

$$H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$$

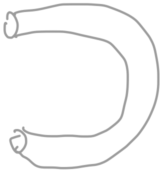
exact

$\square$

Rmk: True for any abelian categories.

Lemma 1.7 (Horseshoe Lemma).

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ short exact sequence.



$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] \arrow[d, hook, "f"] & 0 \\
& & & & B \arrow[d, "g"] & \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & C \arrow[r] & 0
\end{tikzcd}$$

proj. resol'n (free)

proj. resol'n (free)

Then $\exists$ projective resolutions R_* of B with $R_n = P_n \oplus Q_n$ (free)

s.t. right-hand column $A \xrightarrow{f} B \xrightarrow{g} C$ lifts ~~to~~ an exact sequence of complexes

$$0 \to P_* \xrightarrow{i} P_* \oplus Q_* \xrightarrow{\pi} Q_* \to 0$$

s.t $i_n : P_n \hookrightarrow P_n \oplus Q_n$, $\pi_n : P_n \oplus Q_n \to Q_n$ are the natural projections.

Proof:

$$\begin{tikzcd}
Q_0 \arrow[r, dashed, "\exists \tilde{\eta}"] \arrow[dr, "\eta"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

Define $\xi := \varepsilon \oplus \tilde{\eta} : P_0 \oplus Q_0 \longrightarrow B$.

$$\begin{tikzcd}
& & 0 \arrow[d] & & & \\
P_1 \arrow[r, two heads] \arrow[d, hook] & & \mathrm{Ker}(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, "i"] & A \arrow[d, hook, "f"] \\
P_1 \oplus Q_1 \arrow[rr, dashed, "\exists ?"] \arrow[d, two heads] & & \mathrm{Ker}(\xi) \arrow[r, hook] \arrow[d] & P_0 \oplus Q_0 \arrow[r, dashed, "\exists \xi"] \arrow[d, "\pi"] & B \arrow[d, two heads, "g"] \\
Q_1 \arrow[rr, two heads] & & \mathrm{Ker}(\eta) \arrow[r] \arrow[d] & Q_0 \arrow[r, two heads, "\eta"] & C \\
& & 0 & &
\end{tikzcd}$$

Snake's lemma (×3) : is exact

Inductively, we are done. $\square$

Proof of Prop. 1.4 : using Lem. 1.7.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
& F'_* \arrow[u] & F_* \arrow[u] & F''_* \arrow[u] &
\end{tikzcd}$$

get $0 \to F'_* \to F_* \to F''_* \to 0$ s.e.s. of resol'ns.

$\Rightarrow$ $0 \to F'_* \otimes N \to F_* \otimes N \to F''_* \otimes N \to 0$. s.e.s. of chain complexes.

Thm 1.6 $\Rightarrow$ long exact sequence of homologies. $\square$

Example / Exercise 1.8: $R = \mathbb{Z}$. For any $0 \neq m \in \mathbb{Z}$.

Using the free resolution $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ of $\mathbb{Z}/m\mathbb{Z}$.

Show that for any abelian group B:

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, B) = \begin{cases} B/mB, & i=0. \\ {}_mB = \{b \in B \mid mb = 0\}, & i=1. \\ 0, & i \neq 0, 1. \end{cases}$$

m-torsion point of B.

Similarly, $\mathrm{Ext}^n_R(M,N)$ is the derived functor of $\mathrm{Hom}_R(M,N)$.

It can be defined as follows:

- Take free resolution $F_* \to M$ of M.
- Get a cochain complex

$$\mathrm{Hom}_R(F_0,N) \xrightarrow{\circ d} \mathrm{Hom}_R(F_1,N) \xrightarrow{\circ d} \mathrm{Hom}_R(F_2,N) \to \cdots$$

cohom. degree 0 1 2 3

- $\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}_R(F_*,N), \circ d)$.

The above process shows that $\mathrm{Ext}^i_R(M,N)$ is independent of free resolution of M & $\mathrm{Ext}^i_R(-,N): \mathrm{Mod}_R^{op} \to \mathrm{Mod}_R$ is functorial. Moreover, there is a long exact sequence associated with each s.e.s. $0 \to M' \to M \to M'' \to 0$:

$$0 \to \mathrm{Hom}(M'',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N) \to \mathrm{Ext}^1(M'',N) \to \cdots$$

$\square$

Next week: pure abstraction of derived functors à la Grothendieck's [Tohoku] paper.

Aside*: Ext^1 & group of Extensions

Let $R = \mathbb{Z}$, so that $\mathrm{Mod}_R = \mathrm{Ab}$, the category of abelian groups.

An extension of abelian groups B & A is a s.e.s:

$$\xi: \quad 0 \to A \to E \to B \to 0 \qquad \text{in Ab.}$$

2 extensions ξ & ξ' are equivalent if $\exists$ commutative diagram:

$$\begin{tikzcd}
\xi: \quad 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\cong"] & B \arrow[r] \arrow[d, equal] & 0 \\
\xi': \quad 0 \arrow[r] & A \arrow[r] & E' \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

An extension is split if it is equivalent to $0 \to B \xrightarrow{(0,1)} A \oplus B \to A \to 0.$

Def'n: $\mathrm{Ext}(B,A) :=$ equivalent classes of extensions of B by A.

Exercise: Show that there exists a natural group isom:

$$\mathrm{Ext}(B,A) \cong \mathrm{Ext}^1_{\mathbb{Z}}(B,A).$$

Hint: Let $0 \to C \xrightarrow{f} P \to B \to 0$ be a short exact sequence with P a free abelian group.

Show $\exists$ exact sequence

$$\mathrm{Hom}(P,A) \xrightarrow{f^*} \mathrm{Hom}(C,A) \xrightarrow{\partial} \mathrm{Ext}^1(B,A) \to 0$$

$\mathrm{Ext} \Rightarrow \mathrm{Ext}^1$: for any extension ξ, construct $\beta \in \mathrm{Hom}(C,A)$, well-defined up to $f^*\mathrm{Hom}(P,A)$.

$\mathrm{Ext}^1 \Rightarrow \mathrm{Ext}$: for any $\beta: C \to A$, construct an extension ξ as

pushout
$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
\xi \quad 0 \arrow[r] & A \arrow[r] & E \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

Group structure: see Homework.

$\square$