Beauty is the first test, there is no permanent place in the world for ugly mathematics

MATH6502/7502/LN/WZ/ML/2024-2025

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF HONG KONG

MATH6502/7502 Topics in Discrete Applied Mathematics

Part I. Linear Algebra Methods in Discrete Mathematics

1° Rank Argument

This method is to prove a statement by using the rank of a certain matrix, or equivalently, by showing the independence of certain vectors.

(1.1) Counting Clubs in Oddtown 奇偶镇

Suppose that n citizens of oddtown wish to form as many clubs as possible under the following rules:

(a) Each club should have an odd number of members;

(b) Each pair of clubs should have an even number of members in common.

How many clubs can there be in oddtown?

Remark. It is possible to have n clubs in oddtown, for instance, each individual could form a one-member club. The real surprise might be that although n clubs can be formed in a large number of different ways, there is no way of forming $n+1$ or more clubs.

Theorem 1. *In a town of n citizens, no more than n clubs can be formed under rules (a) and (b).*

First Proof. For simplicity we assume that the citizens are numbered 1 through n. Suppose we have m clubs $c_1, c_2, \ldots, c_m$ satisfying (a) and (b). The incidence vector v_i of c_i is a $0-1$ vector with n entries such that the j^{th} entry of v_i is 1 if citizen $j \in c_i$ and 0 otherwise. Then rules (a) and (b) can be rephrased as

forms a m×m real matrix ←

$$v_i^T v_j = \begin{cases} 1 \pmod 2 & \text{if} \quad i = j; \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{1}$$

Then $v_1, v_2, \ldots, v_m$ are linearly independent in F_2^n over the field F_2. To see this, let $\sum_{j=1}^{m} \lambda_j v_j = 0$ be an arbitrary linear relation over F_2 among the v_j's, where $\lambda_j \in F_2$. Then $v_i^T(\sum_{j=1}^{m} \lambda_j v_j) = 0$ for $i = 1, 2, \ldots, m$. By (1), we have $\lambda_i v_i^T v_i = 0$, implying $\lambda_i = 0$, and so the desired statement follows. Hence $m \leq \dim(F_2^n) = n$.

Second Proof. We resort to the following rank inequality:

note: $[n] := \{1 \le i \le n,\ i \in \mathbb{N}_+\}$

Odd - even town: (a), (b) satisfied

n club C_i, $1 \le i \le n$

1. $C_i = \{i\}$, $1 \le i \le n$
2. if n is even, set $C_i = [n] / \{i\} = \{1, 2, \cdots, n\} - \{i\}, \forall i, 1 \le i \le n$

$$|C_i \cap C_j| = \left| [n] / \{i\} \cup \{j\} \right| = 0 \pmod 2$$

Linear algebraic method in Combinatorics (odd - even town)

Theorem 1: $\mathcal{J}$ is a collection of sets, $\mathcal{J} \subseteq 2^{[n]}$ satisfying

1. $(\forall \mathcal{A} \in \mathcal{J}) \to \mathrm{card}(\mathcal{A})$ is odd
2. $(\forall \mathcal{A}, \mathcal{B} \in \mathcal{J}) \wedge (\mathcal{A} \neq \mathcal{B}) \to \mathrm{card}(\mathcal{A} \cap \mathcal{B})$ even

so we have $\mathrm{card}(\mathcal{J}) \le n$

Even - even town question

n citizens in a town, form clubs satisfying

a': each club should have an even number of members

b': each pair of clubs should have an even number of members in common

Theorem 2: $\mathcal{J}$ is a collection of sets, $\mathcal{J} \subseteq 2^{[n]}$ satisfying

1. $(\forall \mathcal{A} \in \mathcal{J}) \rightarrow card(\mathcal{A})$ is even
2. $(\forall \mathcal{A}, \mathcal{B} \in \mathcal{J}) \wedge (\mathcal{A} \neq \mathcal{B}) \rightarrow card(\mathcal{A} \cap \mathcal{B})$ even

so we have $\boxed{card(\mathcal{J}) \ ?}$

$m = card(\mathcal{J})$, $\mathcal{J} = \{C_1, C_2 \cdots\cdots C_m\}$, incidence vectors $v_1 \cdots\cdots v_n$

$(1) \longrightarrow (1)'$ $\quad v_i^T \cdot v_j = 0, \ \forall i, j \in [1, m] \cap \mathbb{Z}$ $\quad$ over $F_2 = \mathbb{Z}/_2$

consider the inner product $\longrightarrow \langle v_i, v_j \rangle = 0$, $v_i \perp v_j \ \forall i, j$

assume S is a collection of some certain vectors v_i

$S = \{v_i\}_{i \in \mathcal{J}}$, according to $(1)' \longrightarrow S \perp S \longrightarrow S \subseteq S^\perp$

U is a linear subspace, spanned by vectors in S

$(\alpha \in U)$: $\quad \alpha = \sum_{i \in \mathcal{J}} \mu_i \cdot v_i$, $\quad v_j \cdot \alpha = \sum_{i \in \mathcal{J}} \mu_i \cdot v_i v_j = 0$

$(\beta \in U)$: $\quad \beta = \sum_{k \in \mathcal{J}} \lambda_k v_k$, $\quad \alpha \cdot \beta = \sum_{i \in \mathcal{J}} \sum_{k \in \mathcal{J}} \mu_i \lambda_k \cdot v_i v_k = 0$

$\forall \alpha, \beta \in U \longrightarrow \alpha \perp \beta \longrightarrow U \perp U \longrightarrow U \subseteq U^\perp$

lemma: $\dim V = n$, $U \subseteq V$ is a linear subspace

$V = U \oplus U^\perp$, $\dim V = n = \dim U + \dim U^\perp$

$n \geqslant 2 \dim U \longrightarrow \dim U \leqslant \lceil \frac{n}{2} \rceil$, $m = card(\mathcal{J}) = |U| \leqslant 2^{\lceil \frac{n}{2} \rceil}$

Linear algebraic method in Combinatorics (even - odd town)

Theorem 2: $\mathcal{J}$ is a collection of sets, $\mathcal{J} \subseteq 2^{[n]}$ satisfying

1. $(\forall \mathcal{A} \in \mathcal{J}) \rightarrow \text{card}(\mathcal{A})$ is even

2. $(\forall \mathcal{A}, \mathcal{B} \in \mathcal{J}) \wedge (\mathcal{A} \neq \mathcal{B}) \rightarrow \text{card}(\mathcal{A} \cap \mathcal{B})$ odd

so we have $\boxed{\text{card}(\mathcal{J}) \ ?}$

lemma: such $\mathcal{J}$ satisfies $\text{card}(\mathcal{J}) \leq n+1$, $\mathcal{J} = \{C_1, C_2 \cdots C_m\}$

adding a new element $n+1$ to every set $C_i \in \mathcal{J}$, so we get a new collection of sets $\mathcal{J}^*$, we know that $\mathcal{J}^*$ satisfying (1)

so we have $\text{card}(\mathcal{J}) = \text{card}(\mathcal{J}^*) \leq n+1$ (theorem 1)

suffice to prove $\text{card}(\mathcal{J}) \neq n+1 \longrightarrow \text{card}(\mathcal{J}) \leq n$

suppose for a contradiction that $\mathcal{J} = \{C_1, C_2 \cdots C_{n+1}\}$

$\forall 1 \leq i \leq n+1$, for each set C_i, we have a vector v_i as before

$n+1$ vectors in a $\dim = n$ $(\mathbb{F}_2^n)$ linear space, so they must be linearly dependent. therefore, there exist $a_i \in \mathbb{F}_2 = \mathbb{Z}/2$ for $1 \leq i \leq n+1$ which are not all 0's such that $\sum_{i=1}^{n+1} a_i v_i = 0$

$$v_i^T v_j = \begin{cases} 0, & \forall i = j \\ 1, & i \neq j \end{cases}$$

we do the same like "Oddtown" $\forall i \in [1; n+1] \cap \mathbb{Z}$

$$0 = v_i^T \cdot \sum_{j=1}^{n+1} a_j v_j = \sum_{j=1}^{n+1} a_j v_i^T v_j = \sum_{j=1}^{n+1} a_j - a_i$$

$$a_i = \sum_{j=1}^{n+1} a_j \quad \forall i \longrightarrow a_1 = a_2 = \cdots\cdots = a_{n+1} = \sum_{j=1}^{n+1} a_j$$

1. $\forall i : a_i = 0$ contradiction

2. $a_1 = a_2 = \cdots\cdots = a_{n+1} = 1$, over field $\mathbb{F}_2$. n must be even

$$\sum_{j=1}^{n+1} v_j = 0$$

consider $J^c = \left\{ C_i^c ,\ C_i \in J \right\}$. we will see that J^c also satisfies even - odd town (theorem 3)

1. $\mathrm{card}(C_i^c) = n - \mathrm{card}(C_i)$ is even, because n is even

2. $\mathrm{card}(C_i^c \cap C_j^c) = \mathrm{card}\left((C_i \cup C_j)^c\right) = n - \mathrm{card}(C_i \cup C_j)$ is odd

$$= n - \mathrm{card}(C_i) - \mathrm{card}(C_j) + \mathrm{card}(C_i \cap C_j)$$

so by the same proof, we have

$\sum_{j=1}^{n+1} v_j' = 0$, where v_j' is the incidence vector for C_j^c

$$0 = \sum_{j=1}^{n+1} v_j + \sum_{j=1}^{n+1} v_j' = \sum_{j=1}^{n+1} \mathbb{1} = (n+1)\,\mathbb{1} = \mathbb{1}$$

$A \in \mathbb{R}^{m\times n}$, $rank(A) = rank(A \cdot A^T)$

Let $M = (v_1 \ v_2 \cdots v_m) \in \mathbb{R}^{n\times m}$

$v_i^T v_j = M^T M$

$$rk(AB) \le \text{Min}\,\{rk(A),\ rk(B)\}, \tag{2}$$

where A and B are matrices over an arbitrary field, and the number of columns of A equals the number of rows of B.

Let M be the $n \times m$ matrix $(v_1, v_2, \ldots, v_m)$, where v_i is as defined in the first proof, and let $A = M^T M$. Then by (1) we have (over F_2) $A = I$. So $rk(A) = m$. From (2) it follows that

$$m = rk(M^T M) \le \text{Min}\,\{rk(M^T),\ rk(M)\} = rk(M) \le n,$$

remark: the rank of a matrix might be different over different field

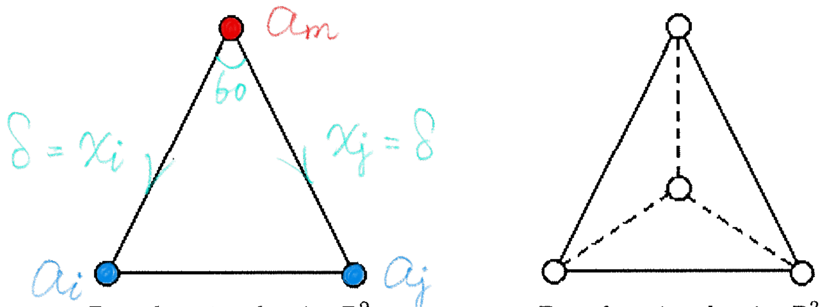
as desired. □

(1.2) Point Sets in R^n with Equal Distance

Theorem 2. *Let $a_1, a_2, \ldots, a_m$ be points in R^n such that the pairwise Euclidean distances of a_i's are all equal. Then $m \le n+1$.*

WTS $m-1$ vector linearly independent $m-1 \le n$ so $m \le n+1$

Remark. The bound is sharp: let $\{a_1, a_2, \ldots, a_m\}$ be the set of vertices of a regular simplex. Then $m = n+1$.



Regular simplex in R^2 Regular simplex in R^3

The following proof is due to Professor M.K. Siu.

Proof. Let $x_i = a_i - a_m$ for $i = 1, 2, \ldots, m-1$, and let the pairwise distance of a_i's be δ. Then $x_i^T x_i = \|a_i - a_m\|^2 = \delta^2$ for any $i \le m-1$. Note that for any $1 \le i \ne j \le m-1$, a_i, a_j, and a_m form an equilateral triangle by hypothesis. So the angle formed by x_i and x_j is $60°$, and thus $x_i^T x_j = \|x_i\| \cdot \|x_j\| \cdot \cos 60° = \delta^2/2$.

WTS We aim to prove that $x_1, x_2, \ldots, x_{m-1}$ are linearly independent, which implies $m-1 \le n$ or $m \le n+1$. We shall actually prove a more general statement.

Proposition. *Let $x_1, x_2, \ldots, x_k \in R^n$ be such that $x_i^T x_i = q$ for any i and $x_i^T x_j = p$ for any $i \ne j$, where p and q are fixed constants with $q > p > 0$. Then $x_1, x_2, \ldots, x_k$ are linearly independent.*

$x_i^T x_i > x_i^T x_j > 0$

We prove by induction on k. The case $k = 1$ is trivial.

$k < n$

concepts

k - simplex: a k - dimensional polytope (多面体) that is the convex hull of its k+1 vertices $u_0 \cdots u_k$

$u_0 \cdots u_k$ affinely independent, $u_1 - u_0 \cdots u_k - u_0$ linearly independent

$$C = \left\{ \sum_{i=0}^{k} \theta_i u_i \,\middle|\, \sum_{i=0}^{k} \theta_i = 1,\ \theta_i \geq 0 \quad \forall i \in [0, k] \cap \mathbb{Z} \right\}$$

regular simplex: simplex and also a regular polytope

a regular k-simplex may be constructed from a regular (k-1)- simplex by connecting a new vertex to all original vertices by the common edge length

$\boxed{\text{question}}$ 在 $\mathbb{R}^n$ Euclid space 中, $e_1\ e_2 \cdots e_n$ 单位正交基

与 $\vec{t} = (t, t, t \cdots t) \in \mathbb{R}^n$ 能否构成 regular simplex

n regular simplex has n+1 points $\begin{cases} (0\,0 \cdots \underset{i}{1} \cdots 0),\ a_i & i \in \overline{1, n} \\ (t\, t \cdots t),\ t \end{cases}$

$$\begin{cases} d(a_i, a_j) = \sqrt{2}, & \forall i \neq j \in \overline{1, n} \\ d(a_i, t) = \sqrt{(t-1)^2 + t^2 \cdot (n-1)} = \sqrt{nt^2 - 2t + 1} \end{cases}$$

$$d(a_i, a_j) = d(a_i, t) \longrightarrow \sqrt{2} = \sqrt{nt^2 - 2t + 1} \longrightarrow nt^2 - 2t - 1 = 0$$

solution: $t_1 = \frac{2 + 2\sqrt{n+1}}{2n} = \frac{1 + \sqrt{n+1}}{n}$, $t_2 = \frac{1 + \sqrt{n+1}}{n}$

$$x_{k+1}^T(\lambda_1 x_1 + \cdots + \lambda_{k+1} x_{k+1}) = 0$$

$$\lambda_1 x_{k+1}^T x_1 + \cdots + \lambda_k x_{k+1}^T x_k + \lambda_{k+1} x_{k+1}^T x_{k+1} = 0$$

$$p \cdot \sum_{i=1}^{k} \lambda_i + \lambda_{k+1} \cdot q = 0 \longrightarrow \lambda_{k+1} = -\frac{p}{q} \sum_{i=1}^{k} \lambda_i$$

Suppose the assertion holds for k. Let us proceed to the case of $k+1$. Consider an arbitrary linear combination $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_{k+1} x_{k+1} = 0$, where $x_i^T x_j = p$ if $i = j$ and q otherwise, with $q > p > 0$. Since WTS $\forall i: \lambda_i = 0$ $x_{k+1}^T(\lambda_1 x_1 + \ldots + \lambda_{k+1} x_{k+1}) = 0$, we get $\lambda_{k+1} = -\frac{p}{q}(\lambda_1 + \lambda_2 + \ldots + \lambda_k)$. So

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k - \frac{p}{q}(\lambda_1 + \cdots + \lambda_k) x_{k+1} = 0$$

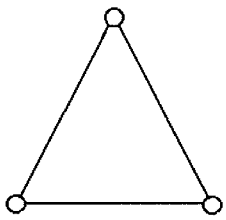
$$\longrightarrow \lambda_1 \underbrace{(x_1 - \frac{p}{q} x_{k+1})}_{y_1} + \lambda_2 \underbrace{(x_2 - \frac{p}{q} x_{k+1})}_{y_2} + \ldots + \lambda_k (x_k - \frac{p}{q} x_{k+1}) = 0.$$

Set $y_i = x_i - \frac{p}{q} x_{k+1}$. Then $y_i^T y_i = q - \frac{p^2}{q} \stackrel{\Delta}{=} q'$ and $y_i^T y_j = p - \frac{p^2}{q} \stackrel{\Delta}{=} p'$. Note that $q' > p' > 0$ as $q > p > 0$. By induction hypothesis with respect to $y_1, y_2, \ldots, y_k$, we have $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$. This, in turn, implies $\lambda_{k+1} = 0$. □

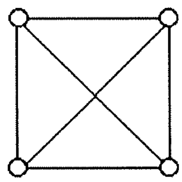
$$y_i^T y_i = (x_i^T - \frac{p}{q} x_{k+1}^T) \cdot (x_i - \frac{p}{q} x_{k+1}) = x_i^T x_i - \frac{p}{q} x_i^T x_{k+1} - \frac{p}{q} x_i x_{k+1}^T + \frac{p^2}{q^2} x_{k+1}^T x_{k+1}$$

(1.3) How Many Complete Bipartite Graphs Add Up to a Complete Graph?

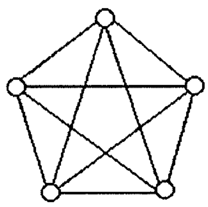
A graph is called *complete* if there is an edge between any two vertices. A complete graph with n vertices is usually denoted by K_n.



K_3

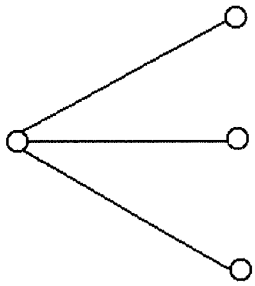


K_4

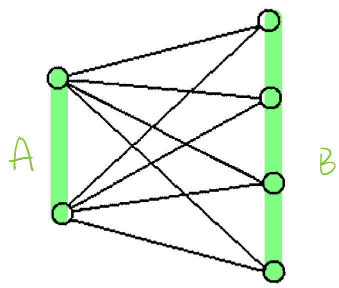


K_5

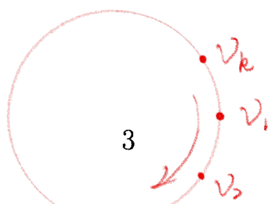
A graph G is *bipartite* if its vertex-set can be partitioned into two (disjoint) subsets A, B such that all edges are between A and B. (No edge is allowed to join two vertices in A or two in B.) We say that (A, B) is the *bipartition* of G. Graph G is called a *complete* bipartite graph if for any $a \in A$ and any $b \in B$, there is an edge between a and b. The complete bipartite graph G is denoted by $K_{s,t}$ if $|A| = s$ and $|B| = t$.



$K_{1,3}$



$K_{2,4}$



cycle $\begin{cases} \text{odd}, & \text{if } k \text{ is odd} \\ \text{even}, & \text{if } k \text{ is even} \end{cases}$

theorem: a graph is bipartite "iff $\Longleftrightarrow$" it contains no odd cycles

The following problem arose in connection with the problem of "addressing into the squashed cube" in telecommunication.

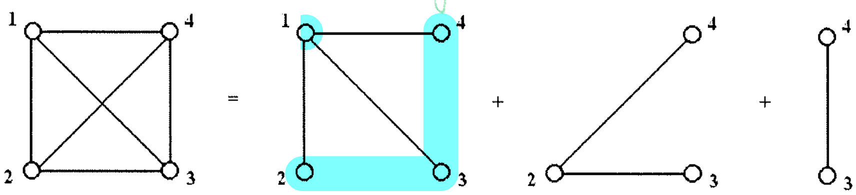
Input K_n – the complete graph on n vertices.

Output Decomposition of K_n into m edge disjoint, complete bipartite graphs such that m is as small as possible.

m个互不相交的完全二分图

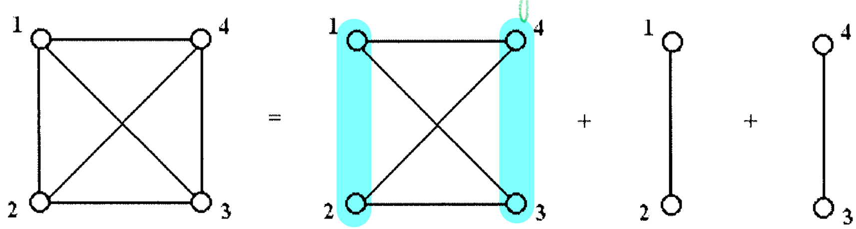
Remark 1. $m \leq n - 1$. For instance,

$K_{1,3}$, 3 edges



Remark 2. There are many decompositions of K_n into $n-1$ edge disjoint complete bipartite graphs. For example,

$K_{2,2}$, 4 edges



The real surprise is $m \geq n - 1$.

Theorem 3 (Graham - Pollak). *If the edge set of K_n is the disjoint union of the edge sets of m complete bipartite graphs, then $m \geq n - 1$.*

First Proof. Suppose K_n has been decomposed into the disjoint union of complete bipartite graphs $B_1, B_2, \ldots, B_m$. Let (X_k, Y_k) be the bipartition of B_k. Now associate an $n \times n$ matrix A_k with each B_k in the following way

Ak 是邻接矩阵的上三角

$$A_k(i,j) = \begin{cases} 1 & \text{if } i \in X_k \text{ and } j \in Y_k; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $rk(A_k) = 1$. Let $S = \sum_{k=1}^{m} A_k$. Then $S + S^T = J - I$, since for each $i \neq j$ precisely one of (i,j) and (j,i) is represented in S, where J is the $n \times n$ all one matrix and I is the $n \times n$ identity matrix.

theorem: decomposition of complete graph K_n into m edges disjoint, complete bipartite grape

see an example $\boxed{K_4 = K_{1,3} + K_3} = K_{1,3} + K_{1,2} + K_{1,1}$

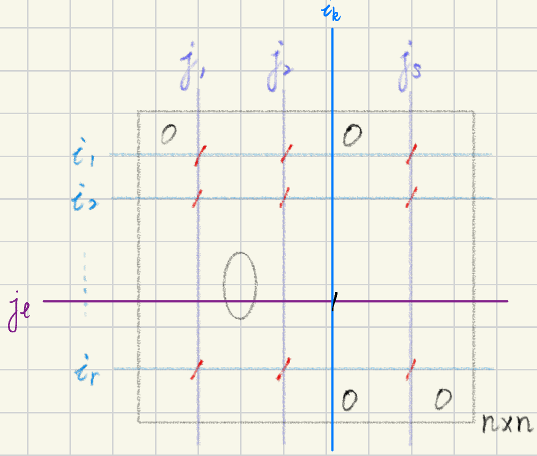
so we have the conclusion: $\boxed{K_n = K_{1,n-1} + K_{n-1}} = K_{1,n-1} + K_{1,n-2} + \cdots + K_{1,1}$

$= K_{1,n-1} + K_{1,n-2} + K_{n-2}$

definition: bipartite graph with its adjacency matrix

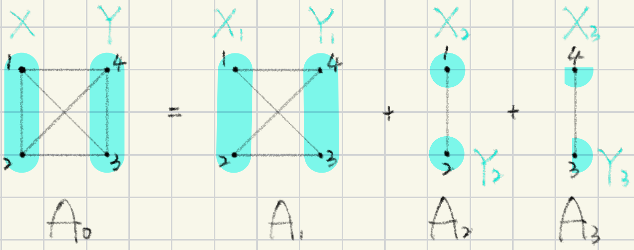
$K_{r,s} = (X, Y)$ where X and Y are two disjoint set including all vertices of $K_{r,s}$.

$X = \{i_1, i_2 \cdots\cdots i_r\}$, $Y = \{j_1, j_2 \cdots\cdots j_s\}$，相应边界处为1



可变换成分块对角矩阵

example



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array} \qquad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A_0 $\qquad$ A_1 $\qquad$ A_2 $\qquad$ A_3

$\qquad$ $A_{2,2}$ $\qquad$ $A_{1,1}$ $\qquad$ $A_{1,1}$

theorem³: Graham - Pollak

如果完全图 K_n 的边集是 m 个完全二分图的边集的不相交并集，则 $m \geq n-1$

1. suppose K_n has been decomposed 分解 into the disjoint union of complete bipartite graphs $B_1, B_2 \cdots B_m$, let (X_k, Y_k) be the bipartition of B_k

associate with an $n \times n$ matrix A_k with each B_k

(X_m, Y_m)

$K_n = B_1 + B_2 + \cdots + B_m$ disjoint

$A_0 = A_1 + A_2 + \cdots + A_m$

from A_k definition, A_k is not symmetric but upper triangular

$$A_k(i,j) = \begin{cases} 1 & \text{if } i \in X_k \text{ and } j \in Y_k \\ 0 & \text{otherwise} \end{cases}$$

A_k's i_1^{th} row $= i_2^{th}$ row, because complete

Clearly, $\boxed{\text{rank}(A_k) = 1}$ Let $S = \sum_{i=1}^{m} A_i \longrightarrow S + S^T = J - I = \begin{bmatrix} 0 & & 1 \\ & 0 & \\ 1 & & \ddots \; 0 \end{bmatrix} = A_0$

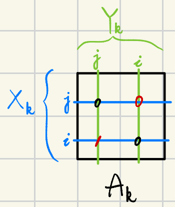
since for each $i \neq j$ precisely one of (i,j) and (j,i) is represented in S (1)

J is the $n \times n$ all one matrix and I is the $n \times n$ identity matrix

$$(1) \longrightarrow \begin{cases} a. & S(i,i) = 0, \quad 1 \leq i \leq n \\ b. & \text{one of } S(i,j) \text{ and } S(j,i) \text{ is } 1, \text{ the other is } 0, \quad 1 \leq i \neq j \leq n \end{cases}$$

proof of a: $X_k \cap Y_k = \emptyset \longrightarrow A_k(i,i) = 0, \forall 1 \leq k \leq m$, and $S = \sum_{k=1}^{m} A_k$

so $S(i,i) = 0$



$i \in X_k$ $j \in Y_k$

$j \notin X_k$ $i \notin Y_k$

not symmetric

proof of b: Let edge $ij \in B_k = (X_k, Y_k)$

either $i \in X_k, j \in Y_k \longrightarrow A_k(i,j) = 1, A_k(j,i) = 0$

or $i \in Y_k, j \in X_k \longrightarrow A_k(i,j) = 0, A_k(j,i) = 1$

$\boxed{\text{Claim.}}$ $\text{rank}(S) \geq n-1$

assuming the claim, it follows that $\overbrace{\boxed{n-1 \leq \text{rank}(S)}}^{\text{WTS}} = \text{rank}\left(\sum_{k=1}^{m} A_k\right) \leq \boxed{\sum_{k=1}^{m} \text{rank}(A_k)} \rightarrow m$

according to $\text{rank}(A_k) = 1, \forall 1 \leq k \leq m$

proof of "Claim" 反证法 $S \in \mathbb{R}^{n\times n}$, $rk(S) \leq n-2$, means $\det(S)=0$ invertible

assume the contrary, $\text{rank}(S) \leq n-2 \longrightarrow$ 矛盾 $\longrightarrow \text{rank}(S) \geq n-1$

then there exists a nonzero solution to the following linear system

$$\boxed{\begin{aligned} &Dx=0,\quad D=\begin{pmatrix} S \\ 1 \end{pmatrix}_{(n+1)\cdot n} \\ &\text{rank}(D)=\text{rank}(S)+1 \leq n-1 \\ &\Omega=\{x\in\mathbb{R}^n : Dx=0\} \\ &\dim(\Omega)=n-\text{rank}(A) \geq 1 \end{aligned}} \longleftarrow \begin{cases} Sx=0, \quad x=(x_1\ x_2 \cdots\cdots x_n)^T & (2) \\ \sum_{i=1}^{n} x_i = 0 \longrightarrow Jx=0 & (3) \end{cases}$$

thus $(S+S^T)x=(J-I)x=-x$, according to (3). From (2) it follows that

$S^Tx=-x$, and hence $-x^Tx = x^TS^Tx=(Sx)^Tx=\overset{\text{means } x=0}{0}$, by (2), contradiction

2. associate with each vertex 顶点 i a variable x_i

$$u_k=\sum_{i\in X_k} x_i \quad \text{and} \quad v_k=\sum_{i\in Y_k} x_i$$

then the fact of decomposition is expressed by equation following

$$\boxed{\sum_{i<j} x_i x_j = \sum_{k=1}^{m} u_k v_k} \quad ? \qquad (4)$$

if $m \leq n-2$, then $\exists$ a nonzero solution $x=(x_1, x_2 \cdots\cdots x_n)^T$ to the following system

$$\begin{cases} u_k=0, \quad k=1,2\cdots\cdots m \\ x_1+x_2+\cdots\cdots+x_n=0 \end{cases}$$

for this solution x

$$\left.\begin{aligned} &LHS \text{ of } (4) = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2-\sum_{i=1}^{n} x_i^2\right] = -\frac{1}{2}\left(\sum_{i=1}^{n} x_i^2\right) < 0 \\ &RHS \text{ of } (4) = 0 \end{aligned}\right\} \text{contradiction}$$

question motivation

1. $\begin{cases} Sx=0 \\ Jx=0 \end{cases}$ 2. $\begin{cases} u_k=0 \\ Jx=0 \end{cases}$

Claim. $rk(S) \geq n-1$.

(Assuming the claim) It follows that $n-1 \leq rk(S) = rk(\sum_{k=1}^{m} A_k) \leq \sum_{k=1}^{m} rk(A_k) = m$, as desired.

Proof of Claim. Assume the contrary: $rk(S) \leq n-2$. Then there exists a nonzero solution to the following linear system

$$\begin{cases} Sx = 0 & (3) \\ \sum_{i=1}^{n} x_i = 0 \quad Jx=0 & (4) \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n)^T$. In view of (4), we have $Jx = 0$. Thus $(S+S^T)x = (J-I)x = -x$. From (3) it follows that $S^T x = -x$, and hence $-x^T x = x^T S^T x = 0$ by (3), a contradiction.

B_k
$\parallel$
(X_k, Y_k)
$\downarrow \quad \downarrow$
$u_k \quad v_k$

Second Proof. Let us associate with each vertex i a variable x_i. Set

consider edge set

$$u_k = \sum_{i \in X_k} x_i \quad \text{and} \quad v_k = \sum_{i \in Y_k} x_i,$$

where X_k and Y_k are as defined in the first proof. Then the fact of the decomposition is expressed by the equation

$$\overset{\text{all edge}}{\boxed{\sum_{i<j} x_i x_j}} = \overset{\text{sum of edges in } B_k}{\sum_{k=1}^{m} u_k v_k} = \sum_{k=1}^{m} \Big(\sum_{i \in X_k} x_i \cdot \sum_{j \in Y_k} x_j\Big) \tag{5}$$

If $m \leq n-2$, then there exists a nonzero solution $x = (x_1, x_2, \ldots, x_n)^T$ to the following linear system $Sx=0$

$$\begin{cases} u_k = 0 \quad \text{for } k = 1, 2, \ldots, m. \\ x_1 + x_2 + \ldots + x_n = 0. \end{cases}$$

For this solution x,

$$\text{LHS of (5)} = \frac{1}{2}[(\sum_{i=1}^{n} x_i)^2 - \sum_{i=1}^{n} x_i^2] = -\frac{1}{2}\sum_{i=1}^{n} x_i^2 < 0,$$
$$\text{RHS of (5)} = 0,$$

a contradiction. □

Proof of Claim 3 Note that

(1) Each edge ij corresponds to $x_i x_j$, and the edge set of B_k corresponds to $u_k v_k$;

(2) $u_k v_k$ and $u_{k'} v_{k'}$ have no common terms (or disjoint) whenever $k \neq k'$, as each $x_i x_j$ appears once in LHS of (5);

(3) Each edge appears in precisely one of B_k's.

(1.4) A Combinatorial Design Problem

One famous block design problem is the following: Maximally how many subsets of a set of size n can pairwise share the same number of elements?

Theorem 4 (Nonuniform Fisher Inequality). *Let $c_1, c_2, \ldots, c_m$ be distinct subsets of a set of size n such that for every $i \neq j$, $|c_i \cap c_j| = \lambda$, where λ is a fixed constant with $1 \leq \lambda < n$. Then $m \leq n$.*

Proof of Claim 1 Let us show that

(a) $S(i,i) = 0 \quad \forall 1 \le i \le n$.

(b) Precisely one of $S(i,j)$ and $S(j,i)$ is 1 and the other is 0 $\forall 1 \le i \ne j \le n$.

$$\Rightarrow S + S^T = J - I = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

(a) Since $A_k(i,i) = 0 \ \forall 1 \le k \le m$, $\Rightarrow S(i,i) = 0$.

(b) Let edge $ij \in B_k$. Then

(1) either $i \in X_k$, $j \in Y_k \Rightarrow A_k(i,j) = 1$, $A_k(j,i) = 0$

or $i \in Y_k$, $j \in X_k \Rightarrow A_k(i,j) = 0$, $A_k(j,i) = 1$.

(2) Since $ij \notin B_\ell \ \forall \ell \ne k$, we have

$$A_\ell(i,j) = A_\ell(j,i) = 0.$$

(1) + (2) $\Rightarrow$ (b) holds $\Rightarrow$ Claim 1 is justified.

Remark Suppose $S + S^T = J - I$ over $\mathbb{F}_2$. It is not necessarily true that $rk(S) \ge n-1$ over $\mathbb{F}_2$, e.g.

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{matrix}$$

$\Rightarrow rk(S) = 3$ over $\mathbb{F}_2$.

Note that $rk(S) < 4$ as $r_1 = r_2 + r_3 + r_4$ over $\mathbb{F}_2$

Proof of Claim 3 Note that

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(2) $u_k v_k$ and $u_{k'} v_{k'}$ have no common terms (or disjoint) whenever $k \ne k'$, as each $x_i x_j$ appears once in LHS of (5);

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(1.4) A Combinatorial Design Problem

One famous block design problem is the following: Maximally how many subsets of a set of size n can pairwise share the same number of elements?

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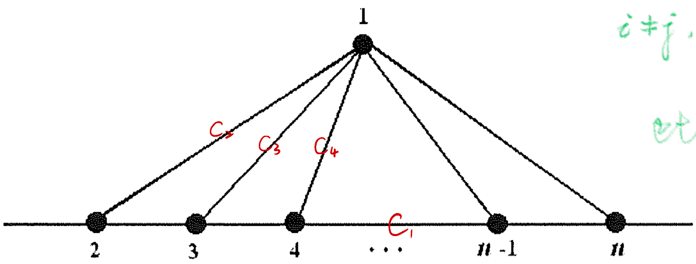
Remark. The bound is sharp.

Example 4.1. Let $c_1 = \{2, 3, \ldots, n\}$ and $c_i = \{1, i\}$ for $i = 2, 3, \ldots, n$. Then $|c_i \cap c_j| = 1$ for every $i \neq j$.

$C_1 \cap C_i = i$

$i \neq j,\ i, j \neq 1,\ C_i \cap C_j = 1$

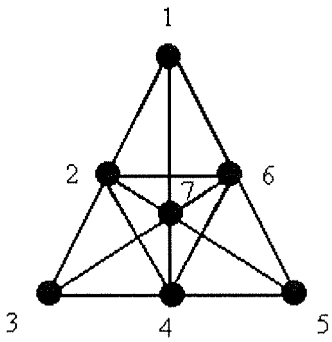
此时 $m = n$



Example 4.2. Let $n = 7$

$$c_1 = \{1, 2, 3\}, \quad c_2 = \{3, 4, 5\}, \quad c_3 = \{5, 6, 1\},$$
$$c_4 = \{1, 7, 4\}, \quad c_5 = \{2, 7, 5\}, \quad c_6 = \{3, 7, 6\},$$
$$c_7 = \{2, 4, 6\}.$$

Then $|c_i \cap c_j| = 1$ for every $i \neq j$.



$C_1\ C_2 \cdots\cdots C_m$ distinct subset

for $i \neq j$ $\quad |C_i \cap C_j| = \lambda$

$j \neq i \neq k$

$|C_k \cap C_j| = \lambda$

C_i

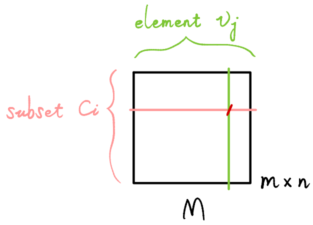
Proof. Let us first consider the case when some c_i has precisely λ elements, that is, $|c_i| = \lambda$. Then $c_i \subseteq c_j$ for any $j \neq i$, and $c_j - c_i$ are pairwise disjoint for all $j \neq i$. Thus $|c_i| + \sum_{j \neq i} |c_j - c_i| \leq n$. So $\lambda + \boxed{m - 1} \leq n$. Hence $m \leq n$ as $\lambda \geq 1$.

$|C_j - C_i| \geq 1$

So we assume hereafter that $|c_i| > \lambda$ for $i = 1, 2, \ldots, m$. Let $v_1, v_2, \ldots, v_n$ be the elements of the set of size n, and let M be the incidence matrix defined as follows

in total n,
$C_1 \cdots\cdots C_m$, m sets
precisely $|C_i| = \lambda$
$n - \lambda \longrightarrow m - 1$

$$M(i, j) = \begin{cases} 1 & \text{if } v_j \in c_i \\ 0 & \text{otherwise.} \end{cases}$$



each row is an incident vector

Notice that M is an $m \times n$ matrix whose rows are indexed by c_i's and columns are indexed by v_j's. Then the intersection condition is summarized in the matrix equation

$\mathrm{rk}(MM^T) \leq \mathrm{rk}(M)$

$$c_i^T c_j = \boxed{MM^T} = \begin{bmatrix} |c_1| & & & \lambda \\ & |c_2| & & \\ & & \ddots & \\ \lambda & & & |c_m| \end{bmatrix} \quad |c_i \cap c_j| = \lambda$$

$$= \lambda J + \mathrm{diag}(|c_1| - \lambda, |c_2| - \lambda, \ldots, |c_m| - \lambda)$$

$$\triangleq \lambda J + \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m),$$

where $\gamma_i = |c_i| - \lambda > 0$ for $i = 1, 2, \ldots, m$.

Claim. MM^T is nonsingular. 非奇异的(det ≠ 0, 可逆) 满秩

(Assuming the claim) It follows that $m = \underbrace{rk(MM^T)}_{m \times m} \leq \underbrace{rk(M)}_{m \times n} \leq n$, as desired. So it remains to establish the above claim.

1. *Method 1*. Show that MM^T is positive definite:

To justify this, note that for any $x = (x_1, x_2, \ldots, x_m)^T \in R^m$, we have

二次型：

$$x^T(MM^T)x = \boxed{\lambda x^T J x} + x^T \mathrm{diag}(\gamma_1, \gamma_2, \ldots \gamma_m)x$$
$$= \boxed{\lambda(\sum_{i=1}^m x_i)^2} + \sum_{i=1}^m \gamma_i x_i^2.$$

$x^T J x = (\sum x_i)^2$

$x^T I x = \sum x_i^2$

Hence $x^T(MM^T)x \geq 0$ for any $x \in R^m$ and equality holds iff $x = 0$.

2. *Method 2*. Show that $\det(MM^T) = \gamma_1\gamma_2 \ldots \gamma_m[1 + \lambda(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_m})]$. Indeed, nonzero

$$\det(MM^T) = \begin{vmatrix} \lambda + \gamma_1 & \lambda & \ldots & \lambda \\ \lambda & \lambda + \gamma_2 & \ldots & \lambda \\ \vdots & \vdots & & \\ \lambda & \lambda & \ldots & \lambda + \gamma_m \end{vmatrix}_{m \times m} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \lambda + \gamma_1 & \lambda & \ldots & \lambda \\ 0 & \lambda & \lambda + \gamma_2 & \ldots & \lambda \\ \vdots & \vdots & \vdots & & \\ 0 & \lambda & \lambda & \ldots & \lambda + \gamma_m \end{vmatrix}_{(m+1)\times(m+1)}$$

每一行减去λ倍第一行

$$\begin{pmatrix} \text{Subtracting } \lambda \text{ times} \\ \text{the } 1^{st} \text{ row from} \\ \text{each of other rows} \end{pmatrix} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ -\lambda & \gamma_1 & 0 & \ldots & 0 \\ -\lambda & 0 & \gamma_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\lambda & 0 & 0 & \ldots & \gamma_m \end{vmatrix}_{(m+1)\times(m+1)}.$$

$$\begin{vmatrix} K & 1 & 1 & \cdots & 1 \\ 0 & \gamma_1 & & 0 & \\ 0 & & \gamma_2 & & \\ \vdots & 0 & & \ddots & \\ 0 & & & & \gamma_m \end{vmatrix}$$

Next, using the diagonal entries, kill the first column to create an upper triangular matrix. The result follows.

det ≠ 0

where $K = \sum_{i=1}^m \frac{\lambda}{\gamma_i} + 1$

the bound in 1.5 is sharp
Let $A_i=\{i\}, \forall 1\le i\le n$
and $A_{n+1}=[n]=\{1,2,3\cdots\cdots n\}$
$\bigcup_{i\in I}A_i=\bigcup_{j\in J}A_j$
$I=\{1,2,3\cdots\cdots n\}$, $J=\{n+1\}$
thus $\bigcap_{i\in I}A_i=\emptyset\ne\bigcap_{j\in J}A_j$
in the 1st case

$\gamma_i=|C_i|-\lambda$

3 *Method 3.* Let x_i be the incidence vector of c_i. Then

MM^T

$$x_i^T x_j = \begin{cases} \lambda+\gamma_i & \text{if } j=i \\ \lambda & \text{if } j\neq i. \end{cases}$$

C_i

pairwise disjoint

Then $x_1, x_2, \ldots, x_m$ are linearly independent.

4 *Method 4.* Compute eigenvalues of MM^T directly and check that all of them are positive.

Exercise. Give a new proof of Theorem 2.

the bound in theorem 5 is sharp:
let $A_i=\{i\}, \forall 1\le i\le n$
$A_{n+1}=[n]:=\{1,2,3\cdots\cdots n\}$
$\bigcup_{i\in I}A_i=\bigcup_{j\in J}A_j$, let $I=\{1,2,3\cdots n\}$
$J=\{n+1\}$ thus $\bigcap_{i\in I}A_i=\emptyset\ne\bigcap_{j\in J}A_j$
in the first case

(1.5) Balanced Families

A family $A_1, A_2, \ldots, A_m$ of distinct sets is *balanced* if there exist two disjoint and nonempty subsets of indices, I and J, such that

$$\bigcup_{i\in I} A_i = \bigcup_{j\in J} A_j \quad \text{and} \quad \bigcap_{i\in I} A_i = \bigcap_{j\in J} A_j.$$

Theorem 5 (Lindstrom). *Every family of m distinct subsets of an n-element set, with $m \geq n+2$, is balanced.*

n个元素的集合的m个不同的子集

Proof. With each subset A of $\{1,2,\ldots,n\}$ we can associate the incidence vector $R^{2n}\ni$ $v=(x_1,y_1,x_2,y_2,\ldots,x_n,y_n)^T$ of the pair $(A,\bar{A})$ in the usual way: $x_i=1$ iff $i\in A$ and $y_i=1-x_i$. These vectors belong to the vector space V (over R) of all vectors for which

if $i\notin A$, $x_i=0$, $y_i=1$

$x_1+y_1=x_2+y_2=\ldots=x_n+y_n$. $=1$ subspace

Claim. The dimension of V is $n+1$.

To prove the claim, observe that for any $v=(x_1,y_1,x_2,y_2,\ldots,x_n,y_n)^T$ in V, the knowledge of $n+1$ coordinates $x_1,x_2,\ldots,x_n,y_n$ is enough to reconstruct the whole vector v, namely $y_i=x_1+y_1-x_i$. So our space V is the set of solutions $v\in R^{2n}$ of the linear system $Mv=0$, where M is the $(n-1)\times(2n)$ matrix

$x_1, y_1, x_2, x_3 \cdots x_n$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots & -1 & -1 \end{pmatrix}_{(n-1)\times(2n)}.$$

So $\dim(V)=2n-rk(M)=2n-(n-1)=n+1$, as desired.

Now let $v_i=(v_{i1},v_{i2},\ldots,v_{i,2n})^T$ be the vector corresponding to the i^{th} set $A_i, i=1,2,\ldots,m$. By the assumption, the vectors $v_1,v_2,\ldots,v_m$ are distinct and all belong to

remark: if $m=n+1$, then $\exists$ two disjoint and nonempty subsets of indices I & J, s.t. $\bigcup_{i\in I}A_i=\bigcup_{j\in J}A_j$. furthermore, $\exists$ two disjoint and nonempty subsets of indices, K & L $\bigcap_{i\in I}A_i=\bigcap_{j\in J}A_j$ But $\nexists$ guarantee that $I=K$ and $J=L$ or $I=L$ and $J=K$

some explaination to the "Claim"

$$V=\{(x_1,y_1\cdots\cdots x_i,y_i\cdots\cdots x_n,y_n)^T\in\mathbb{R}^{2n}\ \text{where}\ x_1+y_1=\cdots=x_i+y_i=\cdots\cdots=x_n+y_n\}$$

$$v\in V \iff \begin{cases} x_1+y_1-x_2-y_2=0 \iff x_1+y_1=x_2+y_2 \\ \vdots \\ x_1+y_1-x_i-y_i=0 \iff x_1+y_1=x_i+y_i \\ \vdots \\ x_1+y_1-x_n-y_n=0 \iff x_1+y_1=x_n+y_n \end{cases} \iff Mv=0, \text{ where}$$

$$M=\begin{array}{c} \begin{array}{cccccccccc} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & & x_n & y_n \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & -1 & -1 \end{pmatrix}_{(n-1)\times(2n)} \end{array}$$

so rank$(M)=n-1$, $\dim(V)=\dim(\mathbb{R}^{2n})-\text{rank}(M)=n+1$ solution space

by assumption, if $m\geq n+2>n+1 \implies v_1\cdots\cdots v_m$ linearly dependent

▲ there exist not all zero $\alpha_i, i\in[m]$ such that $\sum_{i=1}^{m}\alpha_i v_i=0$ $(*)$

Let $I=\{i\mid\alpha_i>0\}$ and $J=\{j\mid\alpha_j<0\}$ then

$I\cap J=\varnothing$, I,J nonempty, $\beta_j=-\alpha_j$, $\forall j\in J$, so the equation $(*)$ is equal to

$$\sum_{i\in I}\alpha_i v_i=\sum_{j\in J}\beta_j v_j$$

while $\alpha_i>0,\ \forall i\in I$ and $\beta_j>0,\ \forall j\in J$

$\forall k\in\bigcup_{i\in I}A_i$

向量v_{i_0}的第$2k-1$个坐标
如果为$2k$，则$k\in A_{i_0}$

$\iff \exists i_0\in I$, such that $k\in A_{i_0}$

$\iff \exists i_0\in I$, such that $\boxed{\underbrace{(2k-1)\text{th entry}}_{2(k-1)+1=2k-1}}$ of v_{i_0} is 1 and $2k^{th}$ entry is 0

$\iff (2k-1)$th entry of $\sum_{i\in I}\alpha_i v_i$ is positive $+$

$\iff (2k-1)$th entry of $\sum_{j\in J}\beta_j v_j$ is positive $+$

$\iff \exists j_0\in J$, such that $(2k-1)$th entry of v_{j_0} is 1

$\iff \exists j_0$ st. $k\in A_{j_0} \iff \forall k\in\bigcup_{j\in J}A_j$

dim $V = n+1$ assume at the beginning

the subspace V. Since $m \geq n+2 > n+1 = \dim(V)$, there must be a nontrivial relation between these vectors, which can be written as

$v_1 \cdots\cdots v_m \in \mathbb{R}^{2n}$

$$\sum_{i \in I} \alpha_i v_i = \sum_{j \in J} \beta_j v_j,$$

where I and J are both nonempty, $I \cap J = \emptyset$, and $\alpha_i, \beta_j > 0$ for all $i \in I$ and $j \in J$. But this means that

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j \quad \text{and} \quad \bigcup_{i \in I} \overline{A_i} = \bigcup_{j \in J} \overline{A_j},$$

where (using the identity $\overline{A} \cup \overline{B} = \overline{A \cap B}$) the last equality amounts to $\bigcap_{i \in I} A_i = \bigcap_{j \in J} A_j$. ☐

De' Morgan

2° Eigenvalue Technique

邻接矩阵

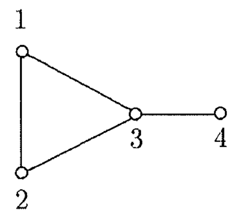
Let G be a graph with n vertices. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix such that

- each row/column is indexed by a vertex;
- the (i, j) entry of $A(G)$ is 1 if vertex i and vertex j are adjacent in G and 0 otherwise.

相邻

For instance,

G :



$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

对称

Let us consider the eigenvalues of $A(G)$.

Example 6.1. Let K_n stand for the complete graph with n vertices. Then the eigenvalues of $A(K_n)$ are

$$n-1, \underbrace{-1, -1, \ldots, -1}_{n-1 \text{ times}}$$

Indeed, $A(K_n) = J - I$. So we have

$$\det(A(K_n) - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 & \ldots & 1 \\ 1 & -\lambda & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \ldots & -\lambda \end{vmatrix}$$
$$= (n-1-\lambda)(-1-\lambda)^{n-1},$$

properties of adjacency matrix 邻接矩阵 A

1. diagonal entries = 0
2. symmetric
3. $A\mathbb{1} = (d_1\ d_2\ \cdots\cdots\ d_n)^T$ where d_i illustrate the degree of vertex i

   and $\mathbb{1}$ is $(1\ 1\ \cdots\cdots\ 1)^T \in \mathbb{R}^n$

4. $A^2(i,j) = \begin{cases} \text{if } i=j, & a_{ij} \text{ express the degree of vertex } i=j \\ \text{if } i\neq j, & \text{the cardinality \# of common neighbers of } i \text{ and } j \end{cases}$

in example 6.1

$$\det(A(K_n) - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 & \dots & 1 \\ 1 & -\lambda & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & -\lambda \end{vmatrix} = (n-1-\lambda)(-1-\lambda)^{n-1},$$

$$\xrightarrow[\text{to the } 1^{st} \text{ row}]{\text{add all rows}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 1 & \cdots & 1 \\ 1 & 1 & -\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & -\lambda \end{vmatrix} \cdot (n-1-\lambda)$$

$$\xrightarrow[\text{from all other rows}]{\text{subtract } 1^{st} \text{ row}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -1-\lambda & 0 & \cdots & 0 \\ 0 & 0 & -1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -1-\lambda \end{vmatrix} \cdot (n-1-\lambda) = (n-1-\lambda)\cdot(-1-\lambda)^{n-1} = 0 \Longrightarrow \begin{cases} 1 \text{ 个 } n-1 \\ n-1 \text{ 个 } -1 \end{cases}$$

some properties:

1. if $\lambda_1\ \lambda_2 \cdots\cdots \lambda_n$ are eigenvalues of $B \in \mathbb{R}^{n\times n}$, then the eigenvalues of $B \pm kI$ are $\lambda_1 \pm k, \lambda_2 \pm k, \cdots\cdots \lambda_n \pm k$, where k is a fixed constant

2. the eigenvalues of J are $n, \underbrace{0, 0 \cdots 0}_{n-1 \text{ times}}$. Furthermore, every eigenvector of J is either praralled or orthogonal 平行或正交 to $\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

3. if $n \geq 3$ and odd $(2 \nmid n)$, then

$$\operatorname{rank}(J - I) = \begin{cases} n, & \text{over } \mathbb{R} \\ n-1, & \text{over } \mathbb{F}_2 \end{cases}$$

if $n \geq 2$ and even $(2 \mid n)$, then

$$\operatorname{rank}(J - I) = n, \text{ over both } \mathbb{R} \text{ \& } \mathbb{F}_2$$

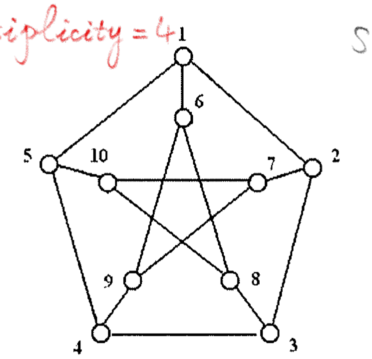
and thus the statement follows. $\square$

Example 6.2. Let G be the Petersen graph. Then the eigenvalues of $A(G)$ are

the eigenvalues of Peterson graph are 3 , 1 , -2 $\longrightarrow$ multiplicity = 4

重数 1，对应向量 $\mathbb{1}$ multiplicity = 5

$$3, 1, 1, 1, 1, 1, -2, -2, -2, -2.$$



Petersen Graph

$$S \cdot T = \begin{bmatrix} -\lambda & 1 & 0 & 0 & 1 \\ 1 & -\lambda & 1 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 0 \\ 0 & 0 & 1 & -\lambda & 1 \\ 1 & 0 & 0 & 1 & -\lambda \end{bmatrix} \cdot \begin{bmatrix} -\lambda & 0 & 1 & 1 & 0 \\ 0 & -\lambda & 0 & 1 & 1 \\ 1 & 0 & -\lambda & 0 & 1 \\ 1 & 1 & 0 & -\lambda & 0 \\ 0 & 1 & 1 & 0 & -\lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^2 & 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & \lambda^2 & 1-\lambda & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & \lambda^2 & 1-\lambda & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2 & 1-\lambda \\ 1-\lambda & 1-\lambda & 1-\lambda & 1-\lambda & \lambda^2 \end{bmatrix}$$

Clearly,

$$|A(G) - \lambda I| = \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left| \begin{array}{ccccc:ccccc} -\lambda & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -\lambda & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -\lambda & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & -\lambda & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\lambda & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -\lambda & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -\lambda \end{array} \right| \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}$$

$$\triangleq \left| \begin{array}{c:c} S & I \\ \hdashline I & T \end{array} \right|.$$

To calculate the determinant, let us apply the following matrix identity

$$\begin{pmatrix} S & I \\ I & T \end{pmatrix} \begin{pmatrix} I & -T \\ 0 & I \end{pmatrix} = \begin{pmatrix} S & I - ST \\ I & 0 \end{pmatrix}.$$

Then we have (by swapping i^{th} column and $(5+i)^{th}$ column for $i = 1, 2, \ldots, 5$).

$$\begin{vmatrix} S & I \\ I & T \end{vmatrix} \begin{vmatrix} I & -T \\ 0 & I \end{vmatrix} = \begin{vmatrix} S & I - ST \\ I & 0 \end{vmatrix} = -\begin{vmatrix} I - ST & S \\ 0 & I \end{vmatrix} = -|I - ST|.$$

$= - \det(I - ST) \cdot \det I$