Lemma (Triangular Criterion). *For $i = 1, 2, \ldots, m$, let $f_i : \Omega \to F$ be a function, where Ω is an arbitrary set and F is a field, and let $a_i \in \Omega$ be such that*

$$f_i(a_j) = \begin{cases} \neq 0 & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{21}$$

Then $f_1, f_2, \ldots, f_m$ are linearly independent over F.

	a_1	a_2	a_3	a_4 ......
f_1	m_1			
f_2	0	m_2		
f_3	0	0	m_3	
f_4	0	0	0	m_4
f_5	0	0	0	0 m_5
$\vdots$				

side. By (21), all but the j^{th} term vanish, and what remains is $\lambda_j f_j(a_j) = 0$. This, again by (21), implies $\lambda_j = 0$, contradicting the choice of j. □

(3.1) Point Sets in R^n with Only Two Distances

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a set of points in R^n. Call A a $\boxed{\textit{two-distance set}}$ if there exist two positive constants δ_1 and δ_2 such that for any $1 \leq i < j \leq m$, the Euclidean distance between a_i and a_j, $\|a_i - a_j\| \in \{\delta_1, \delta_2\}$. Denote

$$m(n) = \text{Max}\{|A| : A \subseteq R^n \quad \text{is a two-distance set}\}.$$

Theorem 11. $n(n-1)/2 \leq m(n) \leq (n+1)(n+4)/2$.

Remark. The ratio of the two bounds tends to 1 as $n \to \infty$. $\frac{n(n-1)}{2} \cdot \frac{2}{(n+1)(n+4)} \xrightarrow{n\to+\infty} 1$

Proof. To establish the lower bound, let e_i be the vector in R^n whose j^{th} entry is 1 if $i = j$ and 0 otherwise for $i = 1, 2, \ldots, n$. Consider

$$A = \{e_i + e_j : 1 \leq i < j \leq n\}.$$

$\left.\begin{matrix} e_a + e_b \\ e_c + e_d \end{matrix}\right\}$ $\begin{matrix} \text{if } a \neq c \neq b \neq d \text{, distance } 2 \\ \text{if } a = c \text{ or } b = d \text{, distance } \sqrt{2} \end{matrix}$

Then the pairwise distances between any two points in A take $\sqrt{2}$ and 2. Hence A is a two-distance set, and therefore $m(n) \geq |A| = n(n-1)/2$. $= \binom{n}{2}$

To derive the upper bound, consider an arbitrary two-distance set $A = \{a_1, a_2, \ldots, a_m\}$, whose two distances are δ_1 and δ_2. We aim to prove that $|A| \leq (n+1)(n+4)/2$.

Define the polynomial

$$F(x, y) = (\|x - y\|^2 - \delta_1^2)(\|x - y\|^2 - \delta_2^2),$$

where $x, y \in R^n$ and $\|\cdot\|$ is the Euclidean norm. Then this polynomial puts our two-distance condition in a simple algebraic form

$$F(a_i, a_j) = \begin{cases} 0 & \text{if } i \neq j; \\ \delta_1^2 \delta_2^2 & \text{otherwise.} \end{cases} \tag{22}$$

Now let $f_i(x) = F(x, a_i)$ for $i = 1, 2, \ldots, m$, where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$. Then (22) becomes

$$f_i(a_j) = \begin{cases} 0 & \text{if } i \neq j; \\ \delta_1^2 \delta_2^2 & \text{otherwise.} \quad i = j \end{cases}$$

So, by the triangular criterion, $f_1(x), f_2(x), \ldots, f_m(x)$ are linearly independent over R.

Claim. All polynomials $f_i(x)$ can be represented as linear combinations of the following ones

$$(\sum_{k=1}^{n} x_k^2)^2,\ (\sum_{k=1}^{n} x_k^2)x_j,\ x_ix_j,\ x_i, 1, \tag{23}$$

where $1 \leq i, j \leq n$.

To justify the claim, let $a_{i,k}$ be the k^{th} entry of a_i. Then

$$\begin{aligned} f_i(x) &= (\|x - a_i\|^2 - \delta_1^2)(\|x - a_i\|^2 - \delta_2^2) \\ &= [\sum_{k=1}^{n} x_k^2 - 2\sum_{k=1}^{n} a_{i,k}x_k + (\sum_{k=1}^{n} a_{i,k}^2 - \delta_1^2)]\cdot \\ &\quad \cdot[\sum_{k=1}^{n} x_k^2 - 2\sum_{k=1}^{n} a_{i,k}x_k + (\sum_{k=1}^{n} a_{i,k}^2 - \delta_2^2)]\,. \end{aligned}$$

Expanding the product, we can see that $f_i(x)$ is a linear combination of terms listed in (23), as claimed.

Now let S be the linear space spanned by polynomials exhibited in (23) over R, that is, S is the set of all polynomials which can be written as

$$\alpha(\sum_{k=1}^{n} x_k^2)^2 + \sum_{j=1}^{n}\beta_j(\sum_{k=1}^{n} x_k^2)x_j + \sum_{1\leq i,j\leq n}\gamma_{ij}x_ix_j + \sum_{i=1}^{n} w_ix_i + c,$$

where $\alpha, \beta_j, \gamma_{ij}, w_i$, and c are all real numbers. Then the dimension of S is at most the number of polynomials exhibited in (23), which is

$$1 + n + \left[\binom{n}{2} + n\right] + n + 1 = (n+1)(n+4)/2.$$

In view of the independence of $f_1(x), f_2(x), \ldots, f_m(x)$, we have $m \leq \dim(S) \leq (n+1)(n+4)/2$, completing the proof. □

(3.2) Sets with Few Intersection Sizes

Let X be a set of n elements, let $\mathcal{F}$ be a family of subsets of X, and let L be a finite set of nonnegative integers. We call $\mathcal{F}$ an *L-intersecting family* if $|A \cap B| \in L$ for every pair of distinct members A, B of $\mathcal{F}$.

Theorem 12 (Frankl-Wilson). *Let L be a set of s integers and let $\mathcal{F}$ be an L-intersecting family of an n-set. Then*

$$|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}.$$

Remark. The bound is sharp in terms of parameters n and s, as shown by the family of all subsets of size $\leq s$ of an n-set.

Proof. Let $L = \{\ell_1, \ell_2, \ldots, \ell_s\}$, $X = \{1, 2, \ldots, n\}$, and $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$, where $A_i \subseteq X$ and $|A_1| \leq |A_2| \leq \ldots \leq |A_m|$. With each set A_i, we associate its incidence

WLOG

for theorem 11 best known upper bound $m(n) \leq \binom{n+2}{2}$

let a_1, a_2 be two vectors in A, then

- case 1 : $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, with $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, then $\| a_1 - a_2 \| = \sqrt{1+1} = \sqrt{2}$

- case 2 $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ then $\| a_1 - a_2 \| = \sqrt{1+1+1+1} = 2$

remark : Blokhuis established the upper bound $m(n) \leq \binom{n+2}{2}$

by showing that $f_1(\bar{x}), f_2(\bar{x}) \cdots\cdots f_m(\bar{x}), x_1, x_2 \cdots\cdots x_n, 1$

are linearly independent in S, where $\bar{x} = (x_1, x_2 \cdots\cdots x_n)$

$$\Longrightarrow m + (n+1) \leq \dim S \leq \frac{(n+1)(n+4)}{2}$$

$$\Longrightarrow m \leq \binom{n+2}{2}$$

in page 23, we define polynomial $f_i(x) = \prod_{k : \ell_k < |A_k|} (v_i^T x - \ell_k)$

substitute $v_j = x$, if $i = j$: $f_i(x) = \prod_{k : \ell_k < |A_k|} (v_i^T v_i - \ell_k) = \prod_{k : \ell_k < |A_k|} (|A_i| - \ell_k) > 0$

if $i \neq j$: $f_i(x) = \prod_{k : \ell_k < |A_k|} (v_i^T v_j - \ell_k) = \prod_{k : \ell_k < |A_k|} (|A_i \cap A_j| - \ell_k) > 0$

note that $|A_i \cap A_j| < |A_i|$ otherwise $|A_i \cap A_j| = |A_i| \Longrightarrow A_i \subseteq A_j \Longrightarrow A_i \subset A_j \Longrightarrow |A_i| < |A_j|$ contradiction the order in assumption

$|A_1| \leq |A_2| \leq |A_3| \leq \cdots\cdots \leq |A_m|$ as $i > j$

$|A_i \cap A_j| = \ell_k$, $\exists\, l_k \in L$ with $l_k < |A_i|$. hence $f_i(v_j) = 0$ with $i \neq j$

in page 23, for example, let $\underline{x} = (x_1, x_2 \cdots\cdots x_n) \in \mathbb{R}^n$

- $f(\underline{x}) = 2x_1^3 x_4 x_7^{17} - x_2^2 x_4^2 x_9$ is mapped into
  $\bar{f}(\underline{x}) = 2x_1 x_4 x_7 - x_2 x_4 x_9$

- $g(\underline{x}) = \underbrace{3x_2^3 x_5^2}_{= 3x_2 x_5} - \underbrace{2x_2^2 x_5^6}_{= 2x_2 x_5}$ is mapped into
  $\bar{g}(\underline{x}) = x_2 x_5$

generators of this space are 1, $x_{i_1} x_{i_2} \cdots\cdots x_{i_k}$ monomial $\binom{s}{k}$ $\forall\, 1 \leq i_1 < i_2 < \cdots\cdots < i_k \leq s$

$x_I := \prod_{i \in I} x_i$, $\forall\, I \subset \{1, 2, 3 \cdots\cdots s\}$, then x_I's are linearly independent over $\mathbb{R}$, because

if $\sum_{I \subset [s]} \lambda_I x_I = 0$, then $\lambda_I = 0$, $\forall\, I \subset [s]$, for otherwise, let $I_0 \subseteq [s]$ be of smallest

size such that $\lambda_{I_0} \neq 0$

setting $x_i = 1$, $\forall\, i \in I_0$ and $x_j = 0$, $\forall\, j \notin I_0$ and plugging it into $\sum_{I \subset [s]} \lambda_I x_I = 0$

we have $\lambda_{I_0} = 0$, a contradiction vector $v_i \in R^n$ such that the k^{th} entry of v_i is 1 if $k \in A_i$ and 0 otherwise. Then $v_i^T v_j = |A_i \cap A_j|$.

For $i = 1, 2, \ldots, m$, let us define the polynomial f_i in n variables as follows

$$f_i(x) = \prod_{k:\, \ell_k < |A_i|} (\underline{v_i^T x} - \ell_k), \text{ where } x \in R^n.$$

when $x = v_i$, $x_i^T \cdot x_i = |A_i|$

Observe that

$$f_i(v_j) = \begin{cases} \neq 0 & \text{if } i = j, \\ 0 & \text{if } i > j. \quad |A_i| \geq |A_j| \end{cases} \tag{24}$$

Let us now restrict the domain of f_i to the n-cube $\Omega = \{0,1\}^n \subseteq R^n$. Since each $v_j \in \Omega$, by (24) and the triangular criterion, f_i, regarded as $\Omega \to R$ functions, are linearly independent over R.

$\forall x \in \Omega$, $\forall n \in \mathbb{N}$, $x_i^n = x_i$

polynomial $p = \sum_I C_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$

so $i_j \in \mathbb{N} \Rightarrow i_j \in \{0,1\}$

$p = \sum_{S \subseteq [n]} a_S \prod_{k \in S} x_k$

In the domain Ω, we have $x_i^2 = x_i$ (over field $\mathbb{F}_2$) for each variable, and thus every $\Omega \to R$ polynomial is multilinear, where a *multilinear polynomial* is the sum of monomials, and each *monomial* is of the form $cx_{i_1}x_{i_2}\ldots x_{i_k}$, where $c \in R$. The *degree* of a multilinear polynomial is the maximum number of variables in a monomial. So each f_i is a multilinear polynomial of degree at most s. Since the space of multilinear polynomials of degree $\leq s$ is generated by the products of $\leq s$ distinct variables, its dimension is $\sum_{k=0}^{s} \binom{n}{k}$, which is an upper bound for m since $f_1, f_2, \ldots, f_m$ are linearly independent. $\square$

consider $f_i(x) = \prod_{\ell \in L} (\underline{v_i}^T \underline{x} - \ell)$

from $\Omega = \{0,1\}^n$ to $\mathbb{F}_p$

Using essentially the same argument, we can also establish the "modular" form of the above theorem. Write $r \in L \pmod p$ if $r \equiv \ell \pmod p$ for at least one $\ell \in L$.

Theorem 13 (Deza-Frankl-Singhi). *Let L be a set of integers and let p be a prime number. Assume $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ is a family of subsets of an n-set such that*

(a) $|A_i| \notin L \pmod p$ *for* $i = 1, 2, \ldots, m$.

(b) $|A_i \cap A_j| \in L \pmod p$ *for* $1 \leq i < j \leq m$.

Then $|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}$. $\square$

4° The General Position Method

Let W be a linear space of dimension n. We say that a point set $S \subseteq W$ is *in general position* if any n points of S are linearly independent. The following set in R^n

$$M_n = \{m_n(\alpha) = (1, \alpha, \ldots, \alpha^{n-1})^T : \alpha \in R\}$$

is called the *moment curve*. Clearly, the points of the moment curve are in general position since

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix} \overset{\text{Vandermonde determinant}}{=} \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i) \ne 0$$

On page 24 Moment curve

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} & \cdots & \alpha_n^{n-1} \end{vmatrix}_{n\times n} \overset{r_i - \alpha_1 \times r_{i-1}}{\underset{\forall 2 \le i \le n}{=\!=}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \cdots & \alpha_n - \alpha_1 \\ 0 & \alpha_2(\alpha_2 - \alpha_1) & \alpha_3(\alpha_3 - \alpha_1) & \cdots & \alpha_n(\alpha_n - \alpha_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \alpha_2^{n-2}(\alpha_2 - \alpha_1) & \alpha_3^{n-2}(\alpha_3 - \alpha_1) & \cdots & \alpha_n^{n-2}(\alpha_n - \alpha_1) \end{vmatrix}_{n\times n}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_2^{n-2} & \alpha_3^{n-2} & \alpha_4^{n-2} & \cdots & \alpha_n^{n-2} \end{vmatrix}_{(n-1)\times(n-1)} \cdot \prod_{2 \le j \le n} (\alpha_j - \alpha_1)$$

$$\overset{IH}{=} \prod_{2 \le i < j \le n} (\alpha_j - \alpha_i) \cdot \prod_{2 \le j \le n} (\alpha_j - \alpha_1) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i).$$

whenever $\alpha_1, \alpha_2, \dots, \alpha_n$ are distinct.

(4.1) Bollabás' Theorem on Set Systems

Theorem 14 (Bollabás). *Let $A_1, A_2, \dots, A_m$ be r-sets and let $B_1, B_2, \dots, B_m$ be s-sets such that*

(a) A_i and B_i are disjoint for $i = 1, 2, \dots, m$;

(b) A_i and B_j intersect whenever $1 \le i \ne j \le m$.

Then $m \le \binom{r+s}{r}$.

Remark. The bound in tight: Let V be an $(r+s)$-set, let $A_1, A_2, \dots, A_m$ be all r-subsets of V, where $m = \binom{r+s}{r}$, and let $B_i = V - A_i$ for each i.

Proof. Let V be the union of all the sets A_i and B_i. We associate a vector $p(v) = (p_0(v), p_1(v), \dots, p_r(v))^T \in R^{r+1}$ with each element $v \in V$ such that the set of vectors obtained are in general position, that is, any $r+1$ of them are linearly independent. With every set $W \subseteq V$ we associate a polynomial $f_W(x)$ in the $r+1$ variables $x = (x_0, x_1, \dots, x_r)^T$ as follows:

$$f_W(x) = \overbrace{\prod_{v \in W} (p_0(v)x_0 + p_1(v)x_1 + \ldots + p_r(v)x_r)}^{\prod_{v\in W} \langle p(v),\, x \rangle}.$$

This is a homogeneous polynomial of degree $|W|$. Clearly,

$$f_W(x) = \begin{cases} \ne 0 & \text{if } x \text{ is orthogonal to } \boxed{\text{none}} \text{ of the } p(v), v \in W; \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Let $f_i(x)$ stand for $f_{B_i}(x)$. Then f_i is homogeneous of degree s.

in general position

The vectors $p(v)$ corresponding to all $v \in A_i$ generate a subspace $U_i \subsetneq R^{r+1}$ of dimension r. Then $\dim(U_i^{\perp}) = r + 1 - \dim(U_i) = 1$. We can therefore select a nonzero $a_i \in U_i^{\perp}$ so that, for any $v \in V$, a_i is orthogonal to $p(v)$ iff $p(v) \in U_i$ iff $v \in A_i$.

From (25) we see that $f_i(a_j) = 0$ precisely when A_j and B_i intersect, that is,

$$\underset{f_{B_i}(a_j)}{f_i(a_j)} = \begin{cases} \ne 0 & \text{if } i = j. \quad A_j \cap B_i = \emptyset \\ 0 & \text{if } i \ne j. \quad A_j \cap B_i \ne \emptyset \end{cases}$$

$$\Downarrow$$

$\exists v \in B_i \subseteq A_j \cap B_i$ such that $p(v) \perp a_j$

- **2-combinations from the set $\{a, b, c\}$ with no repetition allowed:**

$$\{a, b\}, \{b, c\}, \{a, c\}.$$

- **2-combinations from the set $\{a, b, c\}$ with repetition allowed:**

$$\{a, b\}, \{b, c\}, \{a, c\}, \{a, a\}, \{b, b\}, \{c, c\}.$$

Theorem 1. *The total number of r-combinations from an n-set with no repetition allowed is $\binom{n}{r}$.*

Theorem 2. *The total number of r-combinations from an n-set with repetition allowed is $\binom{n+r-1}{r}$.*

From this and the triangular criterion it follows that $f_1, f_2, \ldots, f_m$ are linearly independent. Therefore, their number m is not greater than the dimension of the space of homogeneous polynomials of degree s in $r+1$ variables, which equals the number of s-combinations of $r+1$ distinct objects with repetition, $\binom{(r+1)+s-1}{s}$. So $m \leq \binom{r+s}{s} = \binom{r+s}{r}$, completing the proof. $\square$

(4.2) Gale's Theorem and Kneser's Conjecture

The purpose of this section is to find out how to distribute points fairly evenly on a sphere. An application of this result to the chromatic theory of graphs will be a proof of Kneser's conjecture.

The *r-sphere* $S^r \subset R^{r+1}$ is defined as the set of vectors of unit length in R^{r+1}, that is

$$S^r = \{x \in R^{r+1} : \|x\| = 1\}.$$

An *open hemisphere* centered at a is defined as the part of S^r lying strictly on one side of a linear hyperplane: $\{x \in S^r : a^T x > 0\}$, where $a \in R^{r+1}$ and $a \neq 0$.

Let us consider the problem on how to distribute $2m + r$ points on the r-sphere so that every open hemisphere contains at least m of them. It is easy to fulfill the target on 1-sphere: just take the vertices of a regular $(2m+1)$-gon. How about the general case?

Theorem 15 (Gale). *For every $m, r \geq 0$, there exists an arrangement of $2m + r$ points on the r-sphere such that every open hemisphere contains at least m of them.*

Let us make some preparations before presenting a proof of this theorem.

$M_n = \{m_n(\alpha) = (1, \alpha, \ldots, \alpha^{n-1})^T : \alpha \in R\}$
moment curve

Lemma (Moment Curve Kissing Lemma). *Let $d \geq 1$, $0 \leq k \leq d/2$, and $\alpha_1, \ldots, \alpha_k \in R$. Then there is a linear hyperplane $P : c^T x = 0$ in R^{d+1} such that the moment curve $M_{d+1} \subset R^{d+1}$ lies entirely on one side of P and that the only points of M_{d+1} contained in P are $m_{d+1}(\alpha_i)$, for $i = 1, 2, \ldots, k$.*

Proof. We aim to find a linear hyperplane of the form $c^T x = 0$ for some $c = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T$ in R^{d+1} so that

- $c^T m_{d+1}(\alpha) > 0$ for any $\alpha \in R - \{\alpha_1, \ldots, \alpha_k\}$; and
- $c^T m_{d+1}(\alpha_i) = 0$ for $i = 1, 2, \ldots, k$.

Consider the polynomial $f(\alpha) = \prod_{i=1}^{k} (\alpha - \alpha_i)$. Now define γ_i to be the coefficients of the polynomial f^2, that is,

$$\gamma_0 + \gamma_1 \alpha + \ldots + \gamma_d \alpha^d = [f(\alpha)]^2.$$

This makes sense as $2k \leq d$. With $c = (\gamma_0, \gamma_1, \ldots, \gamma_d)^T$ we have $c^T m_{d+1}(\alpha) = [f(\alpha)]^2$ for any $\alpha \in R$, from which both requirements follow immediately. $\square$

Corollary. *Let $1 \le d \le n-1$ and $0 \le k \le d/2$. Then there exists a matrix $A \in R^{n\times(d+1)}$ with the following properties:*

- *the rows of A are in general position (every $d+1$ of them are linearly independent). Moreover,*
- *for every k-subset I of $[n] = \{1, 2, \dots, n\}$, there exists a vector $c \in R^{d+1}$ such that, letting $Ac = (\beta_1, \beta_2, \dots, \beta_n)^T$, we have $\beta_i = 0$ if $i \in I$ and $\beta_i > 0$ if $i \notin I$.*

Proof. To see this, let A be the matrix with rows $m_{d+1}(\alpha_1)^T, m_{d+1}(\alpha_2)^T, \dots, m_{d+1}(\alpha_n)^T$ for any n distinct reals α_i, and let c be the normal vector of the linear hyperplane corresponding to the set $\{m_{d+1}(\alpha_i) : i \in I\}$ as described in the lemma. □

Now we are ready to prove Gale's theorem. $S^r \subseteq \mathbb{R}^{r+1}$

Proof of Theorem 15. Let $n = 2m+r$. We need to find nonzero vectors $v_1, v_2, \dots, v_n \in R^{r+1}$ such that for every nonzero vector $x \in R^{r+1}$, at least m of the inequalities $v_i^T x > 0$ $(i = 1, 2, \dots, n)$ hold. Indeed, given such v_i, dividing each by its length, we obtain an appropriate set of points on the sphere.

We construct the v_i's in the following somehow mysterious way.

Let $d = 2m - 2$. Take an $n \times (d+1)$ matrix A with the properties guaranteed by the Corollary. Recall that $rk(A) = d+1$. Let $U = \{x \in R^n : A^T x = 0\}$. Then $\dim(U) = n - d - 1$. Now let B be an $n \times (n-d-1)$ matrix whose columns form a basis of U. Denote the rows of B by $v_1^T, v_2^T, \dots, v_n^T$. Since $n - d - 1 = r + 1$, $v_i \in R^{r+1}$ for each i. We aim to prove that $v_1, v_2, \dots, v_n$ are as desired.

$n = 2m+r$

$d = 2m-2$

Since $rk(B) = r+1$ (i.e. B has full column rank), $Bx \neq 0$ for any nonzero x in R^{r+1}. Now let us show that the number of positive entries in Bx is at least m.

Suppose the contrary: $z = (z_1, z_2, \dots, z_n)^T = Bx$ has at most $m-1$ positive entries; let $I \subseteq [n]$ be the corresponding index set of these positive entries. Then $|I| \le m-1 \le d/2$. By the Corollary, there exists $c \in R^{d+1}$ such that $Ac = b = (\beta_1, \beta_2, \dots, \beta_n)^T$ satisfies $\beta_i = 0$ for any $i \in I$ and $\beta_i > 0$ for any $i \notin I$. Since $A^T B = 0$, $b^T z = c^T A^T Bx = 0$. But

$$b^T z = \sum_{i\in I} \beta_i z_i + \sum_{i\notin I} \beta_i z_i = \sum_{i\notin I} \beta_i z_i,$$

so we must have $z_i = 0$ for any $i \notin I$.

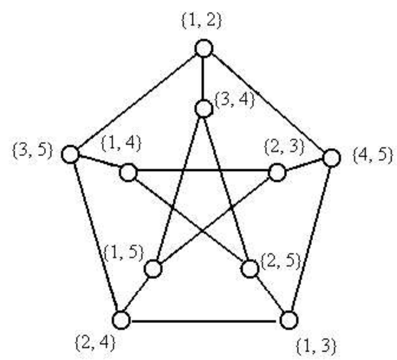
It follows from the above statement that the number of nonzero entries of z is $|I| \le d/2 < d+1$. Since $A^T z = A^T Bx = 0$ and since the rows of A are in general position, we have $z = 0$, contradicting the fact $z = Bx \neq 0$. □

Let us now proceed to Kneser's conjecture and its proof.

In 1955, Kneser proposed a class of graphs with no short odd cycles and with suspected large chromatic number.

Definition. *For $n \geq 2m+1$, the vertex set of Kneser's graph $K(n,m)$ is $\binom{[n]}{m}$, the collection of all m-subsets of $[n] = \{1, 2, \ldots, n\}$. Two vertices $A, B \in \binom{[n]}{m}$ are adjacent iff $A \cap B = \emptyset$.*

Observe that the Peterson graph is the Kneser graph $K(5, 2)$.



Setting $n = 2m+r$ ($r \geq 1$), Kneser's graph has a legal coloring with $r+2$ colors: Take element 1 of $[n]$ and assign color 1 to all m-subsets of $[n]$ containing 1. Next, take element 2 of $[n]$ and assign color 2 to all uncolored m-subsets of $[n]$ containing 2. Proceed in this way until there are $2m-1$ elements of $[n]$ left. We have used $(2m+r)-(2m-1) = r+1$ colors. The $r+2^{nd}$ color is assigned to all m-subsets formed by the remaining $2m-1$ elements (this coloring is valid since any two m-subsets contained in these $2m-1$ elements intersect). A formal description of this procedure is given below.

Algorithm

Step 1. Set $G_1 = K(n,m)$ and $i = 1$.

Step 2. If $i = r+2$, goto Step 4. Else, let V_i be the set of all vertices v of G_i such that the corresponding m-subset of v in $[n]$ contains i. Color all vertices in V_i by color i.

Step 3. Set $G_{i+1} = G_i - V_i$ and $i = i+1$, goto Step 2.

Step 4. Color all vertices of G_{r+2} by color $r+2$, stop.

Kneser conjectured that $r+2$ is the precise chromatic number in all cases. This conjecture was proved by Lovász, using a clever topological method, in 1978.

Theorem 16 (Lovász). *The chromatic number of Kneser's graph $K(2m+r, m)$ is $r+2$.*

The following short proof is due to Bárány, which is based on Gale's theorem on how to distribute points evenly on a sphere (Theorem 15). The following theorem also plays an important role in the proof.

The Antipodal Lemma. *If $U_1, U_2, \ldots, U_{r+1}$ are open sets in S^r that cover S^r, then some U_i contains an antipodal pair of points x, y, that is, $x + y = 0$.* $\square$

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Proof of Theorem 16. Suppose the contrary: $K(2m+r, m)$ can be colored with $r+1$ colors. Let us fix such a coloring and let $G \subset S^r$ be a Gale-set of $2m+r$ points, i.e. a set with the property that every open hemisphere contains at least m points of G (recall Theorem 15). We construct open sets $U_1, U_2, \ldots, U_{r+1}$ in S^r as follows: $x \in U_i$ iff the open hemisphere centered at x contains m points which correspond to a vertex of $K(2m+r, m)$ with color i. Clearly, $U_1, U_2, \ldots, U_{r+1}$ is a cover of S^r. From the antipodal lemma, it follows that some U_k contains a pair of antipodal points x and $-x$, which implies that there is a vertex of $K(2m+r, m)$ with color k contained in the open hemisphere centered at x, there is another vertex of $K(2m+r, m)$ with color k contained in the open hemisphere centered at $-x$. Since these two open hemisphere are disjoint, the two corresponding vertices are adjacent according to the definition of the Kneser graph, but they receive the same color k, a contradiction. $\square$

5° Miscellaneous Topics

(5.1) Helly's Theorems

A *convex combination* of the vectors $v_1, v_2, \ldots, v_m \in R^n$ is a linear combination $\sum_{i=1}^{m} \lambda_i v_i (\lambda_i \in R)$, where $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda_i \geq 0$. A *convex set* is a subset of R^n, closed under convex combinations. The *convex hull* of a subset $S \subseteq R^n$, denoted by $\mathrm{conv}(S)$, is the set of all convex combinations of the finite subsets of S. Clearly, S is convex iff $S = \mathrm{conv}(S)$. It also can be shown that S is convex iff it contains the straight line segment connecting every pair of points of S.

The dimension of a linear space over the reals has characterizations in terms of intersection properties of convex sets. The results, Helly's theorems, have a long history of analogues in combinatorics.

Theorem 17. *If $C_1, C_2, \ldots, C_m \subseteq R^n$ are convex sets such that any $n+1$ of them intersect, then all of them intersect.*

Theorem 18. *If $C_1, C_2, \ldots, C_m, K \subseteq R^n$ are convex sets such that for any $n+1$ of $C_1, C_2, \ldots, C_m$, there exists $d \in R^n$ for which $K + d$ intersects all of them, then there exists $d^* \in R^n$ such that $K + d^*$ intersects all C_i's.*

Remark. The quantity $n+1$ cannot be reduced in either theorem. For Theorem 17, consider the facets of a full-dimensional simplex.

Both theorems can be deduced from the following observation due to Radon.

Lemma (Radon). *Let S be a set of $m \geq n+2$ points in R^n. Then S has two disjoint subsets S_1 and S_2 whose convex hulls intersect.*

Proof. Let $S = \{s_1, s_2, \ldots, \overset{m \geq n+2}{s_m}\}$. Then $s_1 - s_m, s_2 - s_m, \ldots, s_{m-1} - s_m$ are linearly dependent as $m \geq n+2$. So there exist not all zero λ_i's such that $\boxed{\sum_{i=1}^{m-1} \lambda_i (s_i - s_m) = 0,}$ or $\sum_{i=1}^{m-1} (\lambda_i s_i) + \lambda_m s_m = 0$, where $\lambda_m = -\sum_{i=1}^{m-1} \lambda_i$. Note that $\sum_{i=1}^{m} \lambda_i = 0$. Now let S_1 consist of those s_i with nonnegative coefficients λ_i and let $S_2 = S - S_1$. Separating the two subsets of terms in the above equation, we obtain a relation

maybe $\forall j: s_j \in S_2$

we say $\mu_j = -\lambda_j$

$$\sum_{s_i \in S_1} \lambda_i s_i = \sum_{s_j \in S_2} \mu_j s_j, \qquad (26)$$

LHS : conv(S_1)

RHS : conv(S_2)

where $\sum_{s_i \in S_1} \lambda_i = \sum_{s_j \in S_2} \mu_j$, $\lambda_i \geq 0$, $\mu_j > 0$ for each $s_i \in S_1$ and $s_j \in S_2$, and $S_1 \neq \emptyset \neq S_2$. Thus dividing each side of (26) by $\sum_{s_i \in S_1} \lambda_i$, we obtain a point in conv$(S_1) \cap$ conv(S_2). □

Proof of Theorem 17. Assume first that $m = n+2$. Let a_i be a point in $\cap_{j \neq i} C_j$. Let $S = \{a_1, a_2, \ldots, a_m\}$. By Radon's lemma, S has two disjoint subsets S_1 and S_2 with intersecting convex hulls; let $w \in$ conv$(S_1) \cap$ conv(S_2). We claim that w belongs to all C_i. Indeed, for each i, a_i belongs to at most one of S_1 and S_2. Suppose, say $a_i \notin S_1$. $a_i \in S_2$ Then by the selection of a_i, we see that $S_1 \subseteq S - \{a_i\} \subseteq C_i$. So conv$(S_1) \subseteq C_i$ as C_i is convex. Hence $w \in \overset{\forall i}{C_i}$. This concludes the proof for $m = n+2$.

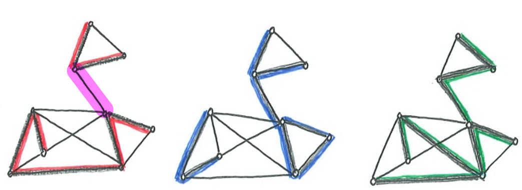
The general case follows by induction on m. For $m \leq n+1$, there is nothing to prove. Assume $m \geq n+3$. By the special case just proved, every $n+2$ of the C_i intersect. So every $n+1$ of $C_1, C_2, \ldots, C_{m-2}, C_{m-1} \cap C_m$ intersect. But then, by induction hypothesis, all intersect. □

The proof of Theorem 18 is left as an exercise.

(5.2) The Matrix-Tree Theorem

Let T be a subgraph of a simple graph G. Call T a *spanning tree* of G if T contains $n-1$ edges and contains no cycles, where n is the number of vertices in G. The purpose of this section is to count the total number of spanning trees, $k(G)$, in a given labeled graph G. For instance, there are 16 distinct spanning trees in labeled K_4 altogether as

(1) Spanning Tree



$n = 9$

edge 8

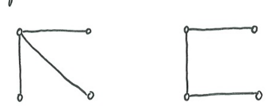
Spanning Tree Counting Problem

Input: A connected, labeled and simple graph G

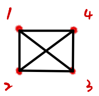
Output: $k(G)$

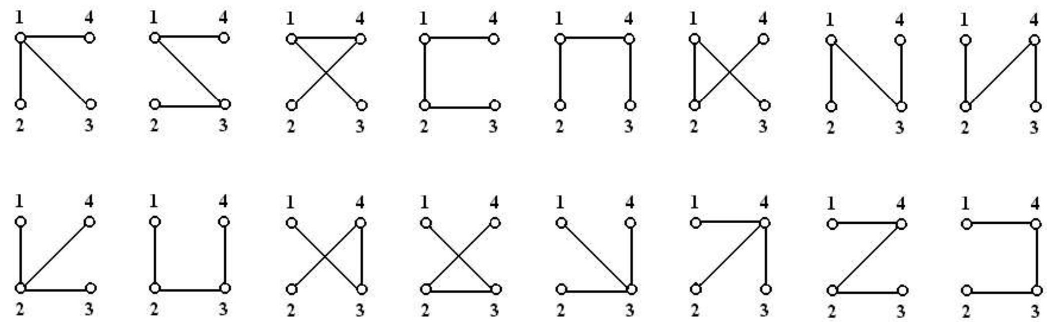
e.g. If K_4 is labeled, then $k(G) = 16$.

If K_4 is unlabeled, then $\exists$ only 2 different spanning trees



shown below. So $k(K_4) = 16$.





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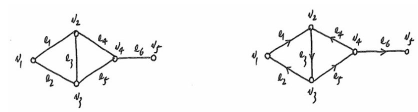
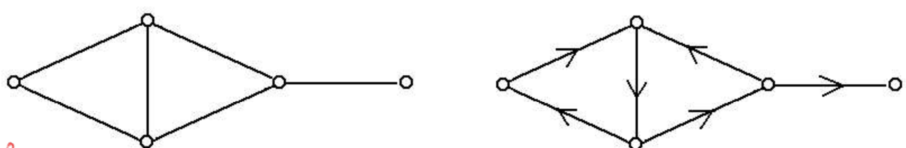
Consider an arbitrary *orientation* D of G, which is a digraph obtained from G by assigning a direction to each edge of G.



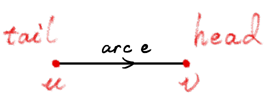
edge

$$A = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix}$$

vertex



G An orientation of G



Each directed edge in D is called an *arc*. If arc e is directed from u to v, then u is called the *tail* and v the *head* of e. Suppose the vertex set of D is $\{v_1, v_2, \ldots, v_n\}$ and the arc set is $\{e_1, e_2, \ldots, e_m\}$. The *incidence matrix* $A = (a_{ij})_{n\times m}$ of D is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } e_j; \\ -1 & \text{if } v_i \text{ is the head of } e_j; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $rk(A) \leq n-1$ as the rows of A sum up to zero. Let A_0 be a matrix obtained from A by removing an arbitrary row. We shall establish the following.

Theorem 19 (The Matrix-Tree Theorem). *The total number of spanning trees in G, $k(G)$, equals* $\det(A_0 A_0^T)$.

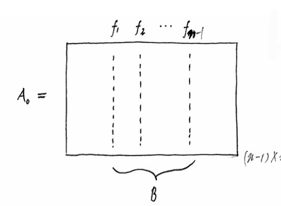
when $n = 2$, $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow \det(A_0 A_0^T) = 1$.

The proof relies heavily on the following lemma.

Lemma 19.1. *Let B be an $(n-1)\times(n-1)$ submatrix of A_0. Then*

$$\det(B) = \begin{cases} \pm 1 & \text{if the spanning subgraph } H \text{ consisting of all the edges} \\ & \text{corresponding to columns in } B \text{ is a spanning tree of } G; \\ 0 & \text{otherwise.} \end{cases}$$



$\Rightarrow H = (V, F)$, where $F = \{f_1, f_2, \ldots, f_{n-1}\}$.
the subgraph corresp. to B.

Proof. Without loss of generality, we assume A_0 is obtained from A by deleting n^{th} row.

Let us prove by induction on n. For $n = 2$, the assertion is trivial. So we proceed to the induction step, and distinguish between two cases.

Case 1. Some v_i, with $i \neq n$, has degree 1 in H.

According to the definition of the incidence matrix, the i^{th} row of B contains exactly one nonzero entry, ± 1, and all other entries in this row belonging to B are zero. Now expand $\det(B)$ by this row. The resulting $(n-2) \times (n-2)$ determinant $\det(B')$ will correspond to $H - v_i$ in the same way as $\det(B)$ does to H, because H is a spanning tree of G iff $H - v_i$ is a spanning tree of $G - v_i$ and $|\det(B)| = |\det(B')|$. The desired statement follows from the induction hypothesis.

▷ *Case 2.* No vertex of H has degree 1, except possibly v_n.

Since H contains $n-1$ edges in total, by the handshaking theorem, there exists a vertex v_j which has degree zero in H. If $v_j \neq v_n$ then the row of B corresponding to v_j is a zero vector, and hence $\det(B) = 0$; if $v_j = v_n$ then each column of B contains a 1 and a -1. So the sum of all rows of B is 0, implying $\det(B) = 0$. □

To prove the theorem, we also need the following well-known formula about the determinant of the product of two matrices.

Binet-Cauchy Theorem. *If Q and R are $k \times m$ and $m \times k$ matrices, respectively, with $k < m$, then the determinant of the product*

$$\det(QR) = \sum_{\substack{I \subseteq \{1,2,\ldots,m\} \\ |I| = k}} \det(Q_I) \cdot \det(R_I),$$

where Q_I (resp. R_I) is the $k \times k$ submatrix of Q (resp. R) determined by the columns (resp. rows) in I. □

Proof of Theorem 19. According to the Binet-Cauchy theorem, we have

$$\det(A_0 A_0^T) = \sum \det(B) \cdot \det(B^T),$$

where B ranges over all $(n-1) \times (n-1)$ submatrices of A_0. From Lemma 19.1 we see

$$\det(B) \cdot \det(B^T) = \begin{cases} 1 & \text{if } B \text{ corresponds to a spanning tree,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $k(G) = \det(A_0 A_0^T)$. □

To facilitate easy computation, let us give an explicit *description* of $A_0 A_0^T$:

$$\text{the } (i,j) \text{ entry of } A_0 A_0^T = \begin{cases} \text{the degree of } v_i \text{ in } G & \text{if } i = j; \\ -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 20 (Cayley). $k(K_n) = n^{n-2}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and let A_0 correspond to $v_1, v_2, \ldots, v_{n-1}$. Then

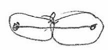
$$A_0A_0^T = \begin{pmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \ldots & n-1 \end{pmatrix}_{(n-1)\times(n-1)} \qquad -J + nI$$

Since $\det(A_0A_0^T) = n^{n-2}$, the statement follows from the matrix-tree theorem. □

in page 31. for case 2

Handshaking Thm Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} d_G(v) = 2|E|.$$



Assume the contrary: $d_H(v_i) \geq 1$ $\forall\, 1 \leq i \leq n$ and $d_H(v_i) \neq 1$ $\forall\, 1 \leq i \leq n-1$. Then $d_H(v_i) \geq 2$ $\forall\, 1 \leq i \leq n-1$.

So

$$2(n-1) = 2|E(H)| = \sum_{i=1}^{n} d_H(v_i) \geq 2(n-1)+1,$$

a contradiction. ■

Binet - Cauchy theorem

$$Q = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \end{bmatrix}_{2\times 3}, \qquad R = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}_{3\times 2}$$

$$QR = \begin{bmatrix} 7 & -1 \\ 1 & 4 \end{bmatrix}$$

- $\det(QR) = 29$.

I: $\{1,2\}$, $\{2,3\}$, $\{3,1\}$

- $\sum_{\substack{I \subseteq [3] \\ |I| = 2}} \det(Q_I) \cdot \det(R_I)$

$$= \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}$$

$= -1 \times 3 + 7 \times 1 + 5 \times 5$

$= 29$.

(i,j) entry of $A_0A_0^T = a_i^T \cdot a_j = \sum_{k=1}^{m} a_{ik} \cdot a_{jk}$ (*)

1. if $i = j$, then $* = \sum_{k=1}^{m} a_{ik} \cdot a_{ik} = \sum_{k=1}^{m} a_{ik}^2$

# of nonzero entries in row i of A_0

2. if $i \neq j$, then $a_{ik} \cdot a_{jk} \neq 0$

if and only if $a_{ik} = 1$, $a_{jk} = -1$ or $a_{ik} = -1$, $a_{jk} = 1$

if and only if $a_{ik} \cdot a_{jk} = -1$

if and only if e_j is an edge between v_i and v_j

since G is a simple graph, $\exists$ at most one edge between vertices v_i and v_j

so $* \neq 0$ if and only if $\exists$ exactly one k with $1 \leq k \leq m$ such that $a_{ik} \cdot a_{jk} = -1$

if and only if $(*) = -1$