$$
\begin{aligned}
&|I-ST| \\
&= (-\lambda^2+4\lambda-3)\cdot\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ \lambda-1 & 1-\lambda^2 & \lambda-1 & \lambda-1 & \lambda-1 \\ \lambda-1 & \lambda-1 & 1-\lambda^2 & \lambda-1 & \lambda-1 \\ \lambda-1 & \lambda-1 & \lambda-1 & 1-\lambda^2 & \lambda-1 \\ \lambda-1 & \lambda-1 & \lambda-1 & \lambda-1 & 1-\lambda^2 \end{vmatrix} \\
&= (-\lambda^2+4\lambda-3)\cdot\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ & 2-\lambda-\lambda^2 & & 0 & \\ & & 2-\lambda-\lambda^2 & & \\ & 0 & & 2-\lambda-\lambda^2 & \\ & & & & 2-\lambda-\lambda^2 \end{vmatrix} \\
&= (-\lambda^2+4\lambda-3)\cdot(2-\lambda-\lambda^2)^4 \\
&= -(\lambda-1)^5(\lambda-3)(\lambda+2)^4
\end{aligned}
$$

It is easy to see that

$$
I - ST = \begin{pmatrix} 1-\lambda^2 & \lambda-1 & \lambda-1 & \lambda-1 & \lambda-1 \\ \lambda-1 & 1-\lambda^2 & \lambda-1 & \lambda-1 & \lambda-1 \\ \lambda-1 & \lambda-1 & 1-\lambda^2 & \lambda-1 & \lambda-1 \\ \lambda-1 & \lambda-1 & \lambda-1 & 1-\lambda^2 & \lambda-1 \\ \lambda-1 & \lambda-1 & \lambda-1 & \lambda-1 & 1-\lambda^2 \end{pmatrix}
$$

So $|A(G)-\lambda I| = -|I-ST| = (\lambda-1)^5(\lambda-3)(\lambda+2)^4$ and thus the eigenvalues of $A(G)$ are as described above. □

def. Peterson Graph: 10个顶点 15条边. 简单图. 每个顶点的 degree = 3

(2.1) Decomposing K_{10} Into Petersen Graphs

	edge
K_{10}	45
Petersen	15

Theorem 6 (Schwenk). *The complete graph on 10 vertices, K_{10}, cannot be expressed as the edge disjoint union of three copies of the Petersen graph.*

The following theorem taken from linear algebra will play an important role in our proof.

Lemma 6.1. *Let $A \in R^{n\times n}$ be a symmetric matrix, let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be all distinct eigenvalues of A, let m_i be the multiplicity of λ_i as an eigenvalue of A, and let $V_i = \{x \in R^n : Ax = \lambda_i x\}$ for $i = 1, 2, \ldots, k$. Then the following statements hold:*

实对称矩阵对应不同特征值的特征向量正交

(a) For any $1 \le i \ne j \le k$ and any $x_i \in V_i$, $x_j \in V_j$, we have $x_i^T x_j = 0$.

(b) $\dim(V_i) = m_i$ for $i = 1, 2, \ldots, k$.

Proof of Lemma 6.1. (a) Notice that $\lambda_i x_i^T x_j = (\lambda_i x_i)^T x_j = (Ax_i)^T x_j = x_i^T(Ax_j) = x_i^T(\lambda_j x_j) = \lambda_j x_i^T x_j$. So $(\lambda_j - \lambda_i)x_i^T x_j = 0$, implying $x_i^T x_j = 0$.

diagonal not Jordan because multiplicity equal to dimension

$$
P^TAP = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}} & & & 0 \\ & \boxed{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ 0 & & & \begin{matrix}\lambda_k & & \\ & \ddots & \\ & & \lambda_k\end{matrix} \end{pmatrix}
\begin{array}{l} \} \; m_1 \text{ times} \\ \} \; m_2 \text{ times} \\ \vdots \\ \} \; m_k \text{ times} \end{array}
$$

$0=\sum_{j=1}^{k} x_j$ when $i \neq j$, $x_i^T x_j = 0$

$0=\sum_{j=1}^{k} x_i^T x_j = x_i^T x_i = |x_i|$

Definition: Let U, W be subspaces of V. Then V is said to be the **direct sum** of U and W, and we write $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$.

Lemma: *Let U, W be subspaces of V. Then $V = U \oplus W$ if and only if for every $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.*

WTS: $\dim(V_i) = m_i$

(b) Suppose $0 = x_1 + x_2 + \ldots + x_k$, where $x_i \in V_i$ for each i. Then we have $x_i^T(x_1 + \ldots + x_k) = 0$ and so $x_i^T x_i = 0$ by (a), which implies $x_i = 0$. It follows that $V_1 + V_2 + \ldots + V_k$ is a direct sum; that is, $V_1 + V_2 + \ldots + V_k = V_1 \oplus V_2 \oplus \ldots \oplus V_k$.

Since A is symmetric, it can be diagonalized as depicted in the above figure by using 正交矩阵P a certain orthogonal matrix P. So the $(m_1 + \ldots + m_{i-1} + 1)^{th}, \ldots, (m_1 + \ldots + m_{i-1} + m_i)^{th}$ columns of P are independent vectors in V_i, implying $\dim(V_i) \geq m_i$. Note that $V_1 \oplus V_2 \oplus \ldots \oplus V_k \subseteq R^n$, we thus have $\dim(V_1 \oplus \ldots \oplus V_k) \leq \dim(R^n) = n$. Hence $n \geq \sum_{i=1}^{k} \dim(V_i) \geq \sum_{i=1}^{k} m_i = n$. It follows that $\dim(V_i) = m_i$ for each i. ☐

Now we are ready to prove Schwenk's theorem.

matrix of Peterson graph is symmetric and $\mathbb{1}$ is eigenvector corresponding 3

Proof of Theorem 6. Since each vertex of the Petersen graph has degree 3, we have

(a) Let A be the adjacency matrix of the Petersen graph. Then $\boxed{A \cdot \mathbf{1} = 3 \cdot \mathbf{1}}$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$.

反证 Suppose to the contrary that K_{10} can be expressed as the edge disjoint union of three Petersen graphs. Let us fix a labeling $1, 2, \ldots, 10$ of the vertices of K_{10}. For this labeling, let A_1, A_2, A_3 be the adjacency matrices of these three copies, respectively. Then

$n=10$, $J = \mathbb{1} \in R^{n\times n}$

$$J - I = A_1 + A_2 + A_3. \qquad (6)$$

adjacency matrix

(b) Let $V_i = \{x \in R^{10} : A_i x = x\}$ be the eigenspace of $\boxed{A_i \text{ corresponding to } 1}$. Then

- V_i is a subspace of $W = \{x \in R^{10} : \mathbf{1}^T x = 0\}$; 与$\mathbb{1}$正交 $\dim W = 10 - 1 = 9$
- $\dim(V_i) =$ the multiplicity of 1 as an eigenvalue $= 5$.

(Indeed, the first statement follows from (a) and Lemma 6.1(a), and the second follows from Example 6.2 and Lemma 6.1(b).)

Since $\dim(W) = 9$, by the dimension formula $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 + V_2) \geq 5 + 5 - \dim(W) = 1$. So there exists a nonzero vector z in $V_1 \cap V_2$. By the definition of V_i and by (b), $A_1 z = z$, $A_2 z = z$, and $\mathbf{1}^T z = 0$. Thus $Jz = 0$ or $(J - I)z = -z$. In view of (6), $(J - I)z = (A_1 + A_2 + A_3)z$, which implies $A_3 z = -3z$. Hence -3 is an eigenvalue of A_3, contradicting Example 6.2. ☐

Our next two examples illustrate the role eigenvalues play in the study of graphs displaying high degree of regularity. We need to start with some more graph terminology.

Let G be a graph. A *cycle* of length k, denoted by C_k, in G is a sequence of k distinct vertices $(a_1, a_2, \ldots, a_k)$ such that a_i is adjacent to $a_{i+1}, 1 \leq i \leq k$, where $a_{k+1} = a_1$. The 周长 *girth* of a graph G is the length of its shortest cycles. The *complement* of G, denoted by $\overline{G}$, is another graph with the same vertex set and complementary edge set: ij is an edge of $\overline{G}$ iff i and j are nonadjacent in G. A vertex v is called a *neighbor* of vertex u if u and

v are adjacent in G. The *degree of u*, denoted by $d(u)$, is the number of all neighbors of u. We say that G is *regular* if all vectices of G have the same degree.

(2.2) The Friendship Theorem

Suppose in a group of people we have the situation that any pair of persons have precisely one common friend. The friendship theorem states that there is always a person (the "politician") who is everybody's friend.

Let us rephrase the theorem in graph-theoretic terms.

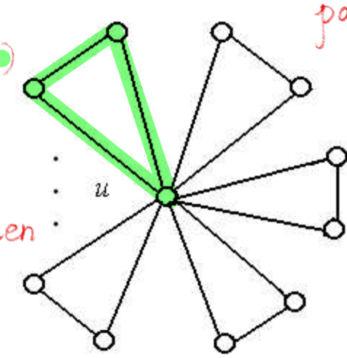
Theorem 7 (The Friendship Theorem). *Suppose that G is a graph in which any two vertices have precisely one common neighbor. Then there is a vertex which is adjacent to all other vertices.*

Note that there are graphs with this property as in the following figure, where u is the politician; in fact we shall show that these "windmill" graphs are the only graphs with this property.

the friendship theorem (equiv. form)

Let G be a graph in which $\exists$ exactly one path of length 2. Then G is a windmill graph

length $= 3 \Longrightarrow$ cycle



A windmill graph

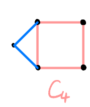
path:

length k

Kotzig's Conjecture

let $l > 3$ then there do NOT exist graphs with the property that between any 2 vertices $\exists$ precisely one path of length l

Proof. Suppose the contrary: no vertex of G is adjacent to all other vertices. To derive a contradiction we proceed in two steps. The first part is combinatorics, and the second part is linear algebra. Observe that the condition of the theorem implies that G contains no C_4. Let us call this the *C_4-condition.*



We claim that

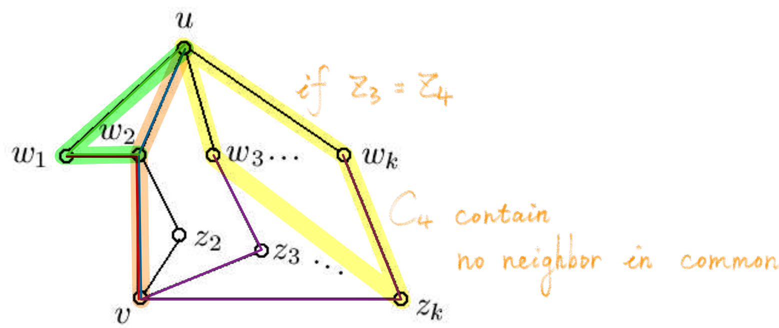
(a) G is a regular graph, that is, $d(u) = d(v)$ for any two vertices u, v in G.

To justify this, we first consider the case when u and v are nonadjacent in G. Suppose $d(u) = k$ and $w_1, w_2, \ldots, w_k$ are the neighbors of u. Since any two vertices have precisely one common neighbor, exactly one of the w_i, say w_2 is adjacent to v (using the pair $\{u, v\}$), and w_2 is adjacent to exactly one of the other w_i's (using the pair $\{u, w_2\}$), say w_1. The vertex v has with w_1 the common neighbor w_2, and with $w_i (i \geq 2)$ a common neighbor $z_i (i \geq 2)$. By the C_4-condition, all these z_i's must be distinct. The situation is

$\forall i, j \in [k]$

w_i, w_j have one common neighbour u

as shown below.



We thus conclude that $d(v) \geq k = d(u)$ and hence $d(u) = d(v) = k$ by symmetry. To finish the proof of (a), observe that any vertex different from w_2 is nonadjacent to u or v, and hence has degree k, by what we have just proved. But since w_2 has a non-neighbor by assumption, it has degree k as well, and thus G is k-regular, so (a) holds.

Let u be a vertex of G and let $w_1, w_2, \ldots, w_k$ be all the neighbors of u. Once again since any two vertices have precisely one neighbor in common, we have

(i) Each w_i has precisely one neighbor in $\{w_1, w_2, \ldots, w_k\}$, and thus has precisely $k-2$ neighbors outside $\{u, w_1, w_2, \ldots, w_k\}$; and   in graph w_1 and w_2

(ii) Each vertex outside $\{u, w_1, \ldots, w_k\}$ is a neighbor of some w_i.   common neighbor of u

Using the C_4-condition, we also have

(iii) The neighbors of $w_1, w_2, \ldots, w_k$ outside $\{u, w_1, \ldots, w_k\}$ are all distinct. u, w_i $w_j \in C_4$, w

Combining (i), (ii) and (iii), we deduce that 推导出

(b) The total number of vertices of G is $n = (k-2)k + k + 1 = k^2 - k + 1$.

Let us now complete the last part of the proof.

Note first that k must be greater than 2, for otherwise only $G = K_1$ and K_3 are possible by (b), both of which are trivial windmill graphs. Consider the adjacency matrix A of G, we obtain

- Each row has exactly k $1'$s by (a), which implies that   regular, degree k

$$A\mathbf{1} = k\mathbf{1} \text{ and so } k \text{ is an eigenvalue of } A. \tag{7}$$

- For any two rows there is exactly one column where they both have a 1, by the condition of the theorem.

Hence

degree of vertex

$$v_i^T \cdot v_j = A^2 = \begin{bmatrix} k & 1 & \ldots & 1 \\ 1 & k & \ldots & 1 \\ & \ldots & & \\ 1 & 1 & \ldots & k \end{bmatrix} = kI + (J - I). \tag{8}$$

incidence vector

K_n

According to Example 6.1, $\boxed{J - I}$ has the eigenvalues $n-1$ (of multiplicity 1) and -1 (of multiplicity $n-1$). So, by (8), A^2 has the eigenvalues $k-1+n = k^2$ (of multiplicity 1) and $k-1$ (of multiplicity $n-1$).

Since A is symmetric, there exists an orthogonal matrix P such that

$$A = P^T \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} P,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all eigenvalues of A. Hence

$$A^2 = P^T \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix} P.$$

Hence each λ_i^2 is an eigenvalue of A^2. So we conclude that

(c) A has the eigenvalues k (of multiplicity 1) and $\pm\sqrt{k-1}$. (In view of (7), k is an eigenvalue.)

Now let r stand for the multiplicity of $\sqrt{k-1}$ and s for the multiplicity of $-\sqrt{k-1}$. Since the sum of the eigenvalues of A equals the trace (which is 0), we find

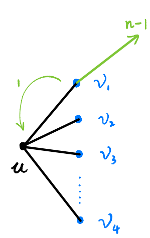
$$k + r \cdot \sqrt{k-1} - s\sqrt{k-1} = 0, \quad (a_{ii} = 0)$$

in particular $r \neq s$, and so $\sqrt{k-1} = k/(s-r)$. It follows that $\sqrt{k-1}$ is an integer h (if $\sqrt{m}$ is rational, then it is an integer! where m is an integer). So

$$h(s-r) = k = h^2 + 1.$$

Since h divides h^2+1 and h^2, we find that h must be equal to 1, and thus $k = 2$, which we have already excluded. So we have reached a contradiction. □

(2.3) Beauty is Rare degree 一致 the length of shortest cycle



Let us consider regular graphs of girth 5, what is the minimum possible number of vertices such a graph can have, if the degree of each vertex is r? + girth 5

Take a vertex u. It has r neighbors. Each of these r neighbors has $r-1$ additional neighbors. We have had $1 + r + r(r-1) = r^2 + 1$ vertices so far, since they must be distinct for otherwise a cycle of length ≤ 4 would arise. It is a natural question to ask: do we need even more vertices? Let us see. For $r = 2$, we have $r^2 + 1 = 5$ vertices, and on page 15

▸ fact: if m is an integer and $\sqrt{m}$ is rational, then $\sqrt{m}$ is an integer

proof: suppose $\sqrt{m} = \frac{q}{p}$ where p and q are relatively prime. by contradiction let $\frac{q}{p}$ is not an integer, then $q^2 = p^2 \cdot m$

Let t^{2k+1} be a maximal prime factors of m, then $t^{2k+1} \mid m \Longrightarrow t^{2k+1} \mid p^2 \cdot m = q^2$

$$t^{2k+1} \mid q^2 \Longrightarrow t^{k+1} \mid q$$

so $p^2 \cdot \left(\frac{m}{t^{2k+1}}\right) = \left(\frac{q}{t^{k+1}}\right)^2 \cdot t \Longrightarrow t \mid p^2 \Longrightarrow t \mid p$. Hence t is a common factor of p and q

contradiction $\square$

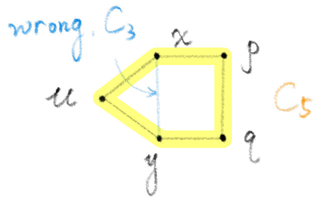
on page 16: principal axis theorem

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalue $\lambda_1 \cdots \lambda_n$. then $\exists$ an orthogonal 正交 matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^T A Q = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

proof: by induction 归纳法 on n. Let u_1 be an eigenvector of A, s.t. $\|u_1\| = 1$

then $\exists$ Q whose 1st column is u_1 $\square$



五边形

there is the pentagon (i.e. C_5). For $r = 3$, we have $r^2 + 1 = 10$ vertices, the Petersen graph is as desired, it can also be shown that the resulting graph is unique.

Can we construct these kinds of graphs for $r \geq 4$?

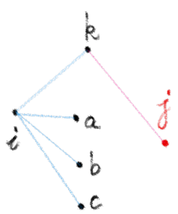
Theorem 8 (Hoffman-Singleton). *If a regular graph of degree r and girth 5 has $r^2 + 1$ vertices, then $r \in \{2, 3, 7, 57\}$.*

Proof. Let G be a regular graph of degree r and girth 5 that has $r^2 + 1$ vertices, let A be its adjacency matrix, and let $\overline{A}$ be the adjacency matrix of $\overline{G}$, the complement of G. Then

$$I + \boxed{A + \overline{A}} = J. \tag{9}$$

contradict with girth 5

For any two vertices i, j of G, observe that i, j have no neighbor in common if ij is an edge of G and i, j have precisely one common neighbor if i, j are nonadjacent in G. To justify this, note that in the former case, we would have a triangle otherwise, contradicting the girth; in the latter case, note that i has r neighbors, each of which has $r - 1$ additional neighbors. We have had $1 + r + r(r - 1) = r^2 + 1$ distinct vertices so far. So there exists a neighbor k of i such that j and k are adjacent. Clearly k is the only common neighbor of i, j, otherwise we would have a cycle of length 4.



Now consider $A^2 = (\beta_{ij})$. Then $\beta_{ii} = r$ and $\beta_{ij} =$ the number of common neighbors of i, j whenever $i \neq j$. According to the statement in the preceding paragraph, $\beta_{ij} = 0$ if ij is an edge of G and 1 otherwise. It follows that

girth 5, no C_3. C_4

$$A^2 = rI + \overline{A} \tag{10}$$

Combining (9) and (10), we obtain

$$A^2 + A - (r - 1)I = J. \tag{11}$$

Since G is r-regular, $A\mathbf{1} = r\mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. So $\mathbf{1}$ is an eigenvector of A corresponding to eigenvalue r. Since A is a symmetric matrix, there exists an orthogonal matrix Q such that $Q^T AQ = \Lambda$, where

- Λ is a diagonal matrix whose diagonal entries are eigenvalues;
- Q consists of eigenvectors of A;
- the first column of Q is $\mathbf{1}/\sqrt{n}$ (so the $(1, 1)$-entry of Λ is r).

Let λ be the eigenvalue of A which is the (i, i)-entry of Λ, where $i \geq 2$, and let e be the i^{th} column of Q. Then $Ae = \lambda e$ and $\boxed{\mathbf{1}^T e = 0}$. Multiplying each side of (11) by e, we get

A symmetric so $Je = \vec{0}$

$$\lambda^2 e + \lambda e - (r - 1)e = 0,$$

implying

$$\lambda^2 + \lambda - (r - 1) = 0.$$

This equation has two roots

theorem: Principal Axis theorem

let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1\ \lambda_2 \cdots \lambda_n$, then $\exists$ an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$Q^T A Q = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$ must can be diagonal instead of Jordan form

proof (Hint): by induction on n

let u_1 be an eigenvector of A corresponding to eigenvalue λ_i with $\|u_i\| = 1$

then $\exists Q$ whose 1^{st} column is u_1

$$\lambda_{1,2} = \frac{1}{2}(-1 \pm \sqrt{4r-3}). \tag{12}$$

Let m_i be the multiplicity of λ_i, $i = 1, 2$. The sum of these multiplicities is $n = r^2 + 1$. So, not forgetting the eigenvalue r, we have

$$1 + m_1 + m_2 = n = r^2 + 1. \tag{13}$$

$m_1 + m_2 = r^2$

Recall that the sum of the eigenvalues is equal to the trace of the matrix, that is, the sum of the diagonal elements. Hence

$$r + m_1\lambda_1 + m_2\lambda_2 = 0. \tag{14}$$

i nonadjacent with i

so $A_{ii} = 0$

Plug (12) into (14), we have

$$2r - (m_1 + m_2) + (m_1 - m_2)s = 0,$$

where $s = \sqrt{4r-3}$. Using (13), this changes to

$$\text{integer} \quad \boxed{2r - r^2} + (m_1 - m_2)s = 0. \tag{15}$$

Notice that s is the square root of a positive integer, so either it is a positive integer, or it is irrational. In the latter case, by (15), $m_1 - m_2$ must vanish, and thus $2r - r^2 = 0$. Since $r \neq 0$, we have $r = 2$, which is the case of the pentagon (i.e. C_5).

We may assume s is a positive integer hereafter. Let us express r through s, i.e. $r = (s^2 + 3)/4$. Plug it into (15), we get

$$s^4 - 2s^2 - 16(m_1 - m_2)s - 15 = 0. \tag{16}$$

Therefore, by (16), s divides 15, and so its possible values are $1, 3, 5$ and 15. From these we obtain the respective values $r = (s^2 + 3)/4 = 1, 3, 7, 57$. We discard the value 1 as $r \geq 2$; the rest is the list of possibilities in addition to $r = 2$. □

Hoffman and Singleton managed to construct a regular graph of girth 5 and degree 7 with 50 vertices. However the case $r = 57$ is still undecided, although computers have been used to aid the search.

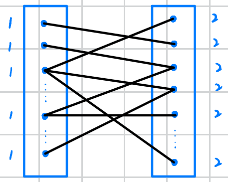
Beautiful graphs are rare, and so are gems like this proof.

(2.4) Eigenvalues and Chromatic Numbers

A graph G is said to be *k-colorable* if there is an assignment of colors, $1, 2, \ldots, k$, to the vertices so that no two adjacent vertices have the same color. The *chromatic number*

Chromatic number of a graph χ

1. let G be a bipartite graph with at least one edge, then $\chi(G) = 2$

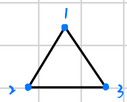


G

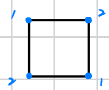
where 1 and 2 means two different colour

2. $\chi(K_n) = n$ complete graph

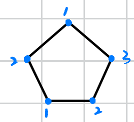
3 $\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$



C_3



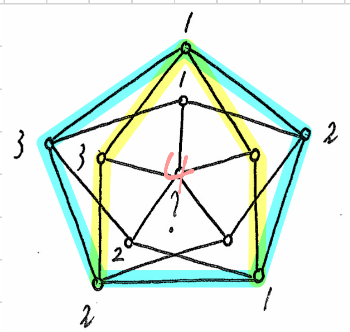
C_4



C_5

where C_n means a cycle graph

let G be the Mycielski graph, the chromatic number $\chi(G) = 4$



proof. $\chi(G) \geq 3$, assume $\chi(G) = 3$

consider a 3-coloring of G and the outer cycle C_5. By symmetry, we may assume that the 3-coloring on this C_5 is as shown above, then ......

Theorem 9. *Let $A(G)$ be the adjacency matrix of a graph G and let λ_1 be the maximum eigenvalue of $A(G)$. Then*

$$\chi(G) \leq 1 + \lambda_1.$$

Theorem 10. *Let G be a graph with n vertices and with at least one edge, let A be a nonzero symmetric $n \times n$ matrix such that $A(i,j) = 0$ whenever i and j are not adjacent, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be eigenvalues of A. Then*

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}. \tag{17}$$

remark. the bounds in theorem 9 & 10 are sharp

Let $G = K_n$ and $A = A(G) = J - I$ then the eigenvalue of A are $n-1$ (of multiplicity 1) and -1 (of multiplicity $n-1$)

so $\lambda_1 = n-1$, $\lambda_2 = -1$, $\chi(K_n) = n \longleftarrow 1 + \lambda_1 = 1 - \frac{\lambda_1}{\lambda_n} = n$

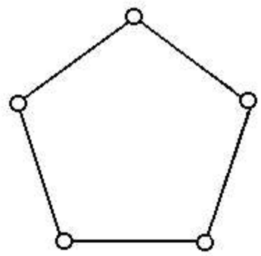
The Interlacing Theorem. *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, let N be an $m \times n$ matrix such that $NN^T = I_m$ (so $m \leq n$), let $B = NAN^T$, and let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ be eigenvalues of B. Then the eigenvalues of B "interlace" those of A; that is*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for} \quad i = 1, 2, \ldots, m. \qquad \square$$

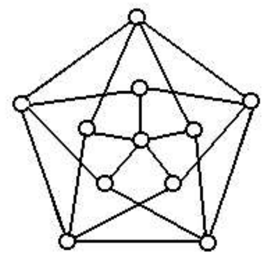
remark. the term "interlacing" seems most natural in the case when $m = n-1$, where

$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots\cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$

of G, denoted by $\chi(G)$, is the smallest k for which G is k-colorable.



$\chi = 3$



$\chi = 4$

Theorem 9. *Let $A(G)$ be the adjacency matrix of a graph G and let λ_1 be the maximum eigenvalue of $A(G)$. Then*

$$\chi(G) \leq 1 + \lambda_1.$$

Theorem 10. *Let G be a graph with n vertices and with at least one edge, let A be a nonzero symmetric $n \times n$ matrix such that $A(i,j) = 0$ whenever i and j are not adjacent, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be eigenvalues of A. Then*

$$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}. \tag{17}$$

To prove the above two theorems, we need the following result.

交错定理 **The Interlacing Theorem.** *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, let N be an $m \times n$ matrix such that $NN^T = I_m$ (so $m \leq n$), let $B = NAN^T$, and let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ be eigenvalues of B. Then the eigenvalues of B "interlace" those of A; that is*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for} \quad i = 1, 2, \ldots, m. \qquad \square$$

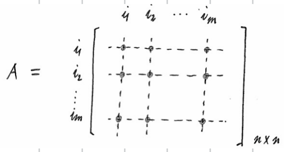
Remark. In particular, the statement holds if B is a symmetric submatrix of A.

Proof of Theorem 9. Since $A(G)$ is symmetric, $\boxed{\lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}}$. Let d denote the minimum degree of G. Then $A(G) \cdot \mathbf{1} \geq d \cdot \mathbf{1}$ (i.e. each entry of $A(G) \cdot \mathbf{1} \geq$ the corresponding entry of $d \cdot \mathbf{1}$) and so $\mathbf{1}^T A(G) \cdot \mathbf{1} \geq d \mathbf{1}^T \mathbf{1}$. Hence

$$\lambda_1 = \max_{x \neq 0} \frac{x^T A(G) x}{x^T x} \geq \frac{\mathbf{1}^T A(G) \mathbf{1}}{\mathbf{1}^T \mathbf{1}} \geq d. \tag{18}$$

Now let us prove the assertion by induction on the number of vertices in G. Let v be a vertex of minimum degree d. In view of (18), $d \leq m - 1$, where $m = \lfloor \lambda_1 \rfloor + 1$. Now consider $G - v$, the graph obtained from G by deleting vertex v. Let λ_1' be the largest eigenvalue of $A(G-v)$. Since $A(G-v)$ is a symmetric submatrix of A, the above remark implies that $\lambda_1' \leq \lambda_1$.

remark, let B be the symmetric supmatrix of A induced by rows $i_1\ i_2 \cdots i_m$ and columns $i_1\ i_2 \cdots i_m$



define

$$N = \begin{array}{c c} & \begin{array}{cccc} i_1 & i_2 & \cdots & i_m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \left[\begin{array}{cccc} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{array}\right]_{m \times n} \end{array}$$

then $B = NAN^T$

定理 6.5 (Courant-Fischer 定理)

令 $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ 是 Hermite 矩阵 $A \in \mathbb{C}^{n \times n}\ (A = A^*)$ 的特征值。那么成立

$$\lambda_k = \max_{L:\dim L = k} \min_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x}, \quad \lambda_k = \min_{L:\dim L = n-k+1} \max_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x}$$

定理 6.6

设 A 是一个 n 级 Hermite 矩阵，而 B 是 A 的 $n-1$ 级顺序（主）子矩阵。那么矩阵 A 的特征值 $\lambda_1 \geqslant \ldots \geqslant \lambda_n$ 和矩阵 B 的特征值 $\mu_1 \geqslant \ldots \geqslant \mu_{n-1}$ 满足划分关系 (соотношение разделения)：

$$\lambda_k \geqslant \mu_k \geqslant \lambda_{k+1}, \quad k = 1, \ldots, n-1$$

证明 令 M 是由向量 $x \in \mathbb{C}^n$ 构成的线性子空间，x 形如

$$x = \begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}, \tilde{x} \in \mathbb{C}^{n-1} \Rightarrow \frac{x^* A x}{x^* x} = \frac{\tilde{x}^* B \tilde{x}}{\tilde{x}^* \tilde{x}}$$

由 Courant-Fischer 定理6.5，可得

$$l\mu_k = \max_{\tilde{L}:\dim \tilde{L} = k} \min_{\tilde{x} \in \tilde{L}, \tilde{x} \neq \theta} \frac{\tilde{x}^* B \tilde{x}}{\tilde{x}^* \tilde{x}} = \max_{L:\dim L = k, L \subseteq M} \min_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x} \leqslant \max_{L:\dim L = k} \min_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x} = \lambda_k$$

$$\mu_k = \max_{\tilde{L}:\dim \tilde{L} = (n-1)-k+1} \min_{\tilde{x} \in \tilde{L}, \tilde{x} \neq \theta} \frac{\tilde{x}^* B \tilde{x}}{\tilde{x}^* \tilde{x}} = \min_{L:\dim L = (n-1)-k+1, L \subseteq M} \max_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x} \geqslant$$

$$\geqslant \min_{L:\dim L = (n-1)-k+1} \max_{x \in L, x \neq \theta} \frac{x^* A x}{x^* x} = \lambda_{k+1}$$

proof of the interlacing theorem

Proof of the Interlacing Thm (Not covered in test / exam)

Let P be an orthogonal matrix s.t. 正交

$$P^T A P = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

and let u_i be the ith column of P. Then

- $\|u_i\| = 1$ and $A u_i = \lambda_i u_i \quad \forall\, 1 \leq i \leq n$
- $u_i^T u_j = 0 \quad \forall\, 1 \leq i \neq j \leq n$.

So $u_i^T A u_j = u_i^T \lambda_j u_j = \lambda_j u_i^T u_j = 0$. We say that $u_1, u_2, \ldots, u_m$ form an orthogonal basis of eigenvectors of A c.t. $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Let $f_1, f_2, \ldots, f_m$ be an orthonormal basis of eigenvectors of $B \in \mathbb{R}^{m \times m}$ c.t. $\mu_1, \mu_2, \ldots, \mu_m$. Fix i with $1 \leq i \leq m$ and consider

$$U = \operatorname{span}\{f_1, f_2, \ldots, f_i\} \subseteq \mathbb{R}^m.$$

For any $x \in U$, write $x = \sum_{j=1}^{i} c_j f_j$, we obtain

$$\frac{\underline{x}^T B \underline{x}}{\underline{x}^T \underline{x}} = \frac{\left(\sum_{j=1}^{i} c_j \underline{f}_j\right)^T B \left(\sum_{j=1}^{i} c_j \underline{f}_j\right)}{\left(\sum_{j=1}^{i} c_j \underline{f}_j\right)^T \left(\sum_{j=1}^{i} c_j \underline{f}_j\right)}$$

$$= \frac{\sum_{j=1}^{i} c_j^2 \underline{f}_j^T B \underline{f}_j}{\sum_{j=1}^{i} c_j^2 \underline{f}_j^T \underline{f}_j}$$

$$= \frac{\sum_{j=1}^{i} c_j^2 \overset{\geq \mu_i}{\mu_j} \underline{f}_j^T \underline{f}_j}{\sum_{j=1}^{i} c_j^2 \underline{f}_j \underline{f}_j} \geq \mu_i$$

Since $\mu_j \geq \mu_i \quad \forall\, 1 \leq j \leq i$, it follows that

$$\frac{\underline{x}^T B \underline{x}}{\underline{x}^T \underline{x}} \geq \mu_i \quad \forall\, \underline{x} \in U. \qquad (1)$$

Let $W = \{ N^T \underline{x} \in \underline{R}^n : \underline{x} \in U \subseteq R^m \}$. Then $N^T \underline{f}_1$, $N^T \underline{f}_2, \ldots, N^T \underline{f}_i$ form a basis of W. So $\dim(W) = i$.

For any $\underline{y} = N^T \underline{x} \in W$, we get

$$\frac{\underline{y}^T A \underline{y}}{\underline{y}^T \underline{y}} = \frac{\underline{x}^T N A N^T \underline{x}}{\underline{x}^T N N^T \underline{x}}$$

$$= \frac{\underline{x}^T B \underline{x}}{\underline{x}^T \underline{x}}.$$

Thus (1) implies

$$\frac{\underline{y}^T A \underline{y}}{\underline{y}^T \underline{y}} \geq \mu_i \quad \forall\, \underline{y} \in W. \qquad (2)$$

Let $V = \operatorname{span}\{\underline{u}_i, \underline{u}_{i+1}, \ldots, \underline{u}_n\} \subseteq R^n$. Then

$$\dim(W \cap V) = \dim(W) + \dim(V) - \dim(W + V)$$

$$\geq i + (n - i + 1) - n = 1.$$

Let $\underline{y} \neq \underline{0}$ be a vector in $W \cap V$. In addition to (2), imitating the proof of (1), we obtain

$$\frac{\underline{y}^T A \underline{y}}{\underline{y}^T \underline{y}} \leq \lambda_i. \qquad (3)$$

Combining (2) and (3), we obtain $\lambda_i \geq \mu_i$.

A similar argument with $U = \operatorname{span}\{\underline{f}_i, \underline{f}_{i+1}, \ldots, \underline{f}_m\}$ will prove that $\mu_i \geq \lambda_{n-m+i}$. ∎

fact: let M be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

by the Principal Axis Theorem, $\exists$ an orthogonal matrix $\mathcal{U}$ s.t. $M = \mathcal{U}^T \begin{pmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \ \lambda_n \end{pmatrix} \mathcal{U}$

so $M = [0]_{n\times n}$, if $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0$

Suppose G admits a proper m-coloring. Let $v_1, v_2, \ldots, v_n$ be an ordering of vertices of G, s.t.

- Vertices of the same color are ordered consecutively;
- Vertices of color i come before any vertices of color j $\forall\, 1 \leq i < j \leq n$.

We may assume that

$$A = A' = \begin{array}{c|c} & v_1 \; v_2 \; \cdots \; v_n \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} & \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix} \end{array}.$$

Otherwise, permute the rows and columns of A in the same way, the resulting matrix has the same eigenvalues as A.

e.g. Permute rows 1 and 2 and permute columns 1 and 2. The resulting matrix

$$A' = \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{bmatrix}$$

$$\hat{=} \; UAU.$$

Note that $UU = I$. So

$$|\lambda I - A'| = |\lambda I - UAU|$$
$$= |U(\lambda I - A)U|$$
$$= |\lambda I - A|.$$

Hence A' has the same eigenvalues as A.

In general, if $B = C^{-1}AC$, then A and B have the same eigenvalues.

proof of theorem 10

Proof of Thm 10 The following statements hold:

(a) $NN^T = I$;

(b) $$N\underset{\sim}{e} = \begin{bmatrix} \|\underset{\sim}{e}_1\| \\ \|\underset{\sim}{e}_2\| \\ \vdots \\ \|\underset{\sim}{e}_m\| \end{bmatrix};$$

(c) $$N^T \begin{bmatrix} \|\underset{\sim}{e}_1\| \\ \|\underset{\sim}{e}_2\| \\ \vdots \\ \|\underset{\sim}{e}_m\| \end{bmatrix} = \underset{\sim}{e}.$$

Justification

(a) NN^T

$$= \begin{bmatrix} \frac{1}{\|\underset{\sim}{e}_1\|}\underset{\sim}{e}_1^T & & & 0 \\ & \frac{1}{\|\underset{\sim}{e}_2\|}\underset{\sim}{e}_2^T & & \\ & & \ddots & \\ 0 & & & \frac{1}{\|\underset{\sim}{e}_m\|}\underset{\sim}{e}_m^T \end{bmatrix}_{m \times n} \cdot \begin{bmatrix} \frac{\underset{\sim}{e}_1}{\|\underset{\sim}{e}_1\|} & & & 0 \\ & \frac{\underset{\sim}{e}_2}{\|\underset{\sim}{e}_2\|} & & \\ & & \ddots & \\ 0 & & & \frac{\underset{\sim}{e}_m}{\|\underset{\sim}{e}_m\|} \end{bmatrix}_{n \times}$$

$$= I.$$

(b) $N\underset{\sim}{e}$

$$= \begin{bmatrix} \frac{1}{\|\underset{\sim}{e}_1\|}\underset{\sim}{e}_1^T & & & 0 \\ & \frac{1}{\|\underset{\sim}{e}_2\|}\underset{\sim}{e}_2^T & & \\ & & \ddots & \\ 0 & & & \frac{1}{\|\underset{\sim}{e}_m\|}\underset{\sim}{e}_m^T \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \vdots \\ \underset{\sim}{e}_m \end{bmatrix}$$

$$= \begin{bmatrix} \|\underset{\sim}{e}_1\| \\ \|\underset{\sim}{e}_2\| \\ \vdots \\ \|\underset{\sim}{e}_m\| \end{bmatrix}.$$

(c) $$N^T \begin{bmatrix} \|\underset{\sim}{e}_1\| \\ \|\underset{\sim}{e}_2\| \\ \vdots \\ \|\underset{\sim}{e}_m\| \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\|\underset{\sim}{e}_1\|}\underset{\sim}{e}_1 & & & 0 \\ & \frac{1}{\|\underset{\sim}{e}_2\|}\underset{\sim}{e}_2 & & \\ & & \ddots & \\ 0 & & & \frac{1}{\|\underset{\sim}{e}_m\|}\underset{\sim}{e}_m \end{bmatrix}_{n \times m} \cdot \begin{bmatrix} \|\underset{\sim}{e}_1\| \\ \|\underset{\sim}{e}_2\| \\ \vdots \\ \|\underset{\sim}{e}_m\| \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \vdots \\ \underset{\sim}{e}_m \end{bmatrix} = \underset{\sim}{e}. \quad \blacksquare$$

$m = \lfloor \lambda_1 \rfloor + 1$

By induction hypothesis $\chi(G-v) \le 1 + \lambda_1' \le 1 + \lambda_1$, so $\chi(G-v) \le m$. Let us color $G-v$ using m colors. Since the degree of v is $d \le m-1$, there exists a color which is not assigned to any neighbor of v; clearly this color is valid for v. Thus $\chi(G) \le m \le 1 + \lambda_1$, as desired. $\square$

$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$

Proof of Theorem 10. From the hypothesis it can be seen that the diagonal entries of A are zero, so the trace of A is zero, and thus $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$. Note that the only symmetric matrix with all its eigenvalues equal to zero is the zero matrix (why?), we have $\lambda_1 > 0 > \lambda_n$. Hence the RHS of (17) is well defined.

Suppose G can be properly colored with m colors. Since A is symmetric, permute the rows and columns in the same way if necessary, we may assume that the color classes induce a partition of A as shown below, where A_{ij} is the submatrix of A consisting of the rows indexed by the vertices of color i and columns indexed by the vertices of color j,

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix}.$$

A_{ij} 块表示
所有 i 颜色的点与
所有 j 颜色的点相邻

Since there is no edge between any two vertices of color i, according to the hypothesis on A, we have $A_{ii} = 0$ for $i = 1, 2, \ldots, m$.

Let e be an eigenvector of A corresponding to λ_1 and write $e = (e_1^T, \ldots, e_m^T)^T$, where e_i has coordinates indexed by the vertices with color i. If none of e_i's is zero, set

$$N = \begin{bmatrix} \frac{1}{\|e_1\|} e_1^T & & & 0 \\ & \frac{1}{\|e_2\|} e_2^T & & \\ & & \ddots & \\ 0 & & & \frac{1}{\|e_m\|} e_m^T \end{bmatrix}_{m \times n} ; \tag{19}$$

$\exists i$ st $e_i = 0$

otherwise, let N be the matrix obtained from the RHS of (19) by deleting all the rows corresponding to $e_i = 0$. Since the latter case goes along the same line as the former, we assume, without loss of generality, that the former case occurs.

Set $B = NAN^T$. Then B is an $m \times m$ matrix whose (i,j)-entry is

$$\frac{1}{\|e_i\|} e_i^T A_{ij} \frac{1}{\|e_j\|} e_j = \frac{1}{\|e_i\| \cdot \|e_j\|} e_i^T A_{ij} e_j.$$

In particular, $B_{ii} = 0$ as $A_{ii} = 0$ for each i. So trace $(B) = 0$.

Note that

$$B(\|e_1\|, \ldots, \|e_m\|)^T = NAN^T(\|e_1\|, \ldots, \|e_m\|)^T = NAe = \lambda_1 Ne = \lambda_1(\|e_1\|, , \ldots, \|e_m\|)^T,$$

so λ_1 is an eigenvalue of B.

$(NN^T)_{ii} = \left(\frac{e_i^T}{\|e_i\|}\right)^2 = 1$

Now let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ be eigenvalues of B. Since $\boxed{NN^T = I}$, by the interlacing theorem, μ_i's interlace eigenvalues of A and hence they are between λ_1 and λ_n. Observe that $\mu_1 = \lambda_1$ according to the statement in the preceding paragraph. So

$$0 = \text{trace}\,(B) = \mu_1 + \mu_2 + \ldots + \mu_m \geq \lambda_1 + (m-1)\lambda_n,$$

completing the proof. $\square$

Let $\omega(G)$ stand for the number of vertices of a largest complete subgraph in G, and let $U(G)$ be the set of symmetric $n \times n$ matrices A such that $A(i,j) = 0$ if i and j are not adjacent in G. Suppose further that $C \subseteq V(G)$ induces a complete subgraph of G. Let A_C denote the $n \times n$ matrix with (i,j)-entry equal to 1 if i and j are two vertices in C and 0 otherwise. Then $A_C \in U(G)$ and it can be shown that $1 - \dfrac{\lambda_1(A_C)}{\lambda_n(A_C)} = |C|$. Using Theorem 10, we thus obtain

$$\chi(G) \geq \max_{A \in U(G)} \left(1 - \frac{\lambda_1(A)}{\lambda_n(A)}\right) \geq \omega(G). \tag{20}$$

Inequality (20) is one of a number of important results obtained by Lovász in his work on the Shannon capacity of a graph; it also serves as the starting point of polynomial-time algorithms for several optimization problems on perfect graphs by Grötschel, Lovász and Schrijver.

3° Polynomial Technique

In order to apply the linear algebra method, in many situations it is particularly useful to associate sets with some multivariate polynomials $f(x_1, x_2, \ldots, x_n)$ (rather than with their incidence vectors) and then show that these polynomials are linearly independent as members of the corresponding functional space. The idea, known as the *polynomial technique*, has many applications in discrete mathematics. Let us present some of them. All these applications are based on the following lemma.

Lemma (Triangular Criterion). *For $i = 1, 2, \ldots, m$, let $f_i : \Omega \to F$ be a function, where Ω is an arbitrary set and F is a field, and let $a_i \in \Omega$ be such that*

$\{a_i\} \subset \Omega$

$$f_i(a_j) = \begin{cases} \neq 0 & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{21}$$

? or $i \neq j$

f_i

F

Then $f_1, f_2, \ldots, f_m$ are linearly independent over F.

Proof. For a contradiction, assume that there exist not all zero $\lambda_i \in F$ such that $\sum_{i=1}^{m} \lambda_i f_i = 0$. Let j be the smallest i with $\lambda_i \neq 0$. Substitute a_j for the variable on each side. By (21), all but the j^{th} term vanish, and what remains is $\lambda_j f_j(a_j) = 0$. This, again by (21), implies $\lambda_j = 0$, contradicting the choice of j. $\square$

$\lambda_j \neq 0$