Notes on algebraic combinatorics

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Abstract

This article consists of some notes taken by the author while studying algebraic combinatorics.

References: [1], [2], [3], [4].

Contents

1 Some motivating results in combinatorics

We start by presenting two results which are special cases of the Heron-Rota-Welsh conjecture and the Dowling-Wilson conjecture.

1.1 The read conjecture

Given a graph G and a positive integer t, a **proper coloring** of G is an assignment of one of t-colors to each vertex of G such that the two endpoints of each edge are assigned different colors. The **chromatic polynomial** of G is defined to be the total number of proper colorings as a function of t, denoted by $P_G(t)$.

Example 1.1.
- *When G has a loop, $P_G(t) = 0$.*

- *When G is a connected tree with n-vertices, $P_G(t) = t(t-1)^{n-1}$.*
- *When G is the complete graph on n vertices, $P_G(t) = t(t-1)\cdots(t-n+1)$.*

The chromatic polynomial satisfies the following **deletion-contraction** relation.

Lemma 1.1 (deletion-contraction)**.** *Given any edge e of G, we have*

$$P_G(t) = P_{G\setminus e}(t) - P_{G/e}(t),$$

where $G \setminus e$ and G/e are the graphs obtained from G by deleting and contracting the edge e, respectively.

Using inductions on the number of edges, one can easily deduce the following properties from the deletion-contraction relation.

Corollary 1.1.
- *For any loopless graph G, the chromatic polynomial $P_G(t)$ is a monic polynomial.*
- *The coefficients of $P_G(t)$ are alternating, that is, if $P_G(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, then $(-1)^i a_i \geq 0$.*

The following result of Huh confirms the Read conjecture from the 60's.

Theorem 1.1 (Huh, 2012)**.** *The (absolute values of the) coefficients of $P_G(t)$ form log-concave sequence. In other words, if $P_G(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, then $|a_{i-1}a_{i+1}| \leq a_i^2$.*

Remark 1.1. *In fact, the sequence $|a_1|, \ldots, |a_n|$ has no internal zeros (which follows from Huh's proof, but is also a previously known result). Obviously, among the nonzero terms, $|a_{i-1}a_{i+1}| \leq a_i^2$ is equivalent to $\log|a_i|$ being concave. For the rest of the note, when we say a sequence a_k is log-concave, we mean the sequence $|a_k|$ has no internal zeros, and $|a_{k-1}a_{k+1}| \leq a_k^2$ for all k.*

1.2 The realizable Dowling-Wilson conjecture

Let V be a d-dimensional vector space over a field, and let $E \subset V$ be a finite generating set. Let

$$\mathcal{F} = \{\text{all linear subspaces of } V \text{ generated by a subset of } E\},$$

and $\mathcal{F}_k = \{F \in \mathcal{F} \mid \dim F = k\}$. Denote the cardinality of $\mathcal{F}_k$ by W_k. The sequence of numbers W_k are called **Whitney numbers of the second kind**. The following result confirms a conjecture of Dowling-Wilson in the realizable case.

Theorem 1.2. *Let W_k be defined as above. For $k \leq d/2$, we have $W_k \leq W_{d-k}$.*

It turns out that for both theorems, the proofs essentially use algebraic geometry. In fact, the first theorem reduces to the Hodge index theorem, and the second reduces to the hard Lefschetz theorem for projective varieties.

Both statements can be generalized to **matroids**, where the corresponding variety may not exist. Then the key idea is to produce some combinatorial proofs of the analogous statements (Hodge index theorem, or more generally the Hodge-Riemann relations, and the hard Lefschetz theorems) on some combinatorial cohomology ring of matroids (or Chow rings, or intersection cohomology groups).

2 Introduction to matroids

2.1 Definations of matroid

Definition 2.1 (independent set). *In graph theory, an independent set, stable set, coclique or anticlique is a set of vertices in a graph, no two of which are adjacent.*

Definition 2.2. *Given a finite set E in a vector space, we can consider the independent sets of E, defined as*

$$\mathcal{I}_E = \{I \subset E \mid I \text{ is an independent set}\}.$$

Definition 2.3. *Given a graph G with edge set $E(G)$, we can define the set of forests,*

$$\mathcal{I}_G = \{I \subset E(G) \mid I \text{ does not contain a cycle}\}.$$

The set of independent sets and forests share a common combinatorial property, called the exchange lemma.

Lemma 2.1 (Steinitz exchange lemma). *Let U and W be finite subsets of a vector space V. If U is a set of linearly independent vectors, and W spans V, then:*

1. $|U| \leq |W|$;
2. *There is a set $W' \subseteq W$ with $|W'| = |W| - |U|$ such that $U \cup W'$ spans V.*

Lemma 2.2 (exchange lemma). *Let $\mathcal{I}_E$ be defined as above. If $I_1, I_2 \in \mathcal{I}_E$ satisfies $|I_2| > |I_1|$, then there exists $x \in I_2$ such that $I_1 \cup x \in \mathcal{I}_E$[1]. The same statement holds for the set of forests $\mathcal{I}_G$.*

Matroid is the combinatorial structure that captures the independence conditions from both linear algebra and graphs.

Definition 2.4 (matroid). *A **matroid** consists of a finite set E and a collection of subsets $\mathcal{I} \subset 2^E$, satisfying the following properties.*

[1] Throughout this note, we abuse notations and use the element x to also denote the singleton set $\{x\}$.

1. $\emptyset \in \mathcal{I}$;

2. if $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$;

3. if $I_1, I_2 \subset \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $x \in I_2$ such that $I_1 \cup x \in \mathcal{I}$.

The sets in $\mathcal{I}$ are called ***independent sets****.*

When E is a finite subset of a vector space, the independent sets $\mathcal{I}_E$ defines a matroid.

There are many other equivalent definitions of matroids in terms of flats, rank functions, closure operators, bases, basis polytopes, etc. Here, we mention a few.

Definition 2.5. *A matroid is a pair $(E, \mathcal{B})$, where E is a finite set and $\mathcal{B} \subset 2^E$, satisfying the following properties.*

1. $\mathcal{B}$ *is nonempty;*

2. *If $A, B \in \mathcal{B}$ and $x \in A \setminus B$, then there is $y \in B \setminus A$ such that $(A \setminus x) \cup y \in \mathcal{B}$.*

The sets in $\mathcal{B}$ are called ***bases****.*

Definition 2.6. *A matroid is a pair $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F} \subset 2^E$, satisfying the following properties.*

1. $\mathcal{F}$ *is nonempty;*

2. *if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;*

3. *given any $F \in \mathcal{F}$, every element of $E \setminus F$ is in exactly one minimal set in $\mathcal{F}$ which strictly contains F.*

The sets in $\mathcal{F}$ are called ***flats****.*

Remark 2.1. *For any pair of flats F, G of a matroid M, there exists a unique minimal flat that contains both F and G, which is called the* ***join*** *of F and G, and denoted by $F \vee G$. There also exists a unique maximal flat that is contained in both F and G, which is called the* ***meet*** *of F and G, and denoted by $F \wedge G$. Clearly, $F \wedge G = F \cap G$, but $F \cup G \subset F \vee G$ and the inclusion can be strict.*

Remark 2.2. *When E is a finite subset of a vector space V, the set of flats*

$$\mathcal{F} = \{W \cap E \mid W \subset V \text{ is a linear subspace}\}$$

defines a matroid.

Definition 2.7. *A matroid is a pair (E, rk), where E is a finite set and $\mathrm{rk} : 2^E \to \mathbb{Z}_{\geq 0}$ satisfying the following properties.*

1. *for any* $S \subset E$, $\mathrm{rk}(S) \leq |S|$*;*

2. *if* $S \subset T \subset E$*, then* $\mathrm{rk}(S) \leq \mathrm{rk}(T)$*;*

3. *the function* rk *is submodular, that is,*

$$\mathrm{rk}(S) + \mathrm{rk}(T) \geq \mathrm{rk}(S \cup T) + \mathrm{rk}(S \cap T)$$

   for any $S, T \subset E$*.*

The function rk *is called the* ***rank function****.*

Remark 2.3. *When* E *is a finite subset of a vector space* V*,* $\mathrm{rk}(S) = \dim \mathrm{span}(S)$ *defines a matroid.*

Definition 2.8. *Let* M *be a matroid defined over a set* E *using one of the above equivalent definitions. Then* E *is called the* ***ground set*** *of* M*. The* ***rank*** *of* M *is defined to be* $\mathrm{rk}(E)$*, and also denoted by* $\mathrm{rk}(M)$*. An element* $i \in E$ *is called a* ***loop*** *if* $\mathrm{rk}(i) = 0$*, which is equivalent to that* i *is not contained in any independent set, and is further equivalent to that* i *is contained in every flat. Two elements* $i, j \in E$ *are called* ***parallel****, if* $\mathrm{rk}(i) = \mathrm{rk}(j) = \mathrm{rk}(\{i, j\}) = 1$*. If a matroid has no loops, then we say it is* ***loopless****. If a matroid has no loops or parallel elements, then we say it is* ***simple****. An element* $i \in E$ *is called a* ***coloop*** *if* $\mathrm{rk}(E \setminus i) = \mathrm{rk}(E) - 1$*, or equivalently,* i *is contained in every basis.*

Example 2.1. *A* ***uniform matroid*** *on* $E = \{1, \ldots, n\}$ *with rank* r*, denoted by* $U_{r,n}$*, is a matroid whose bases are all* r*-element subsets of* E*. When* $r = n$*, the matroid* $U_{n,n}$ *is called a* ***Boolean matroid****.*

Definition 2.9. *A matroid defined using trees in a graph as discussed before is called a* ***graphic matroid****. Given a field* K*, a matroid defined by a finite subset* E *in a* K*-vector space* V *is called* ***realizable over*** K*. A matroid realizable over some field* K *is called* ***realizable****. A finite set* E *in a vector space* V *is called a* ***vector configuration****.*

Proposition 2.1. *A graphic matroid is realizable over any field.A graphic matroid is realizable over any field.*

Proof. Given a finite graph G, we label its vertices by $1, 2, \ldots, n$. Fixing any field K, we let V be the K-vector space with basis $v_1, \ldots, v_n$. Let the set of edges be $E = \{e_1, \ldots, e_m\}$, and assume that the two ends of each e_i be $a_i, b_i \in \{1, \ldots, n\}$. Now, we choose the subset $E_V \subset V$ to be $\{v_{a_i} - v_{b_i} \mid i = 1, \ldots, m\}$. Then it is easy to check that a subset of E form a forest if and only if the corresponding set of vectors in V are independent. $\square$

Remark 2.4. *As in the above proof, the subset* E *does not generate* V*. In fact, the codimension of* $\mathrm{span}(E)$ *is equal to the number of connected components of* G*.*

Example 2.2. *The fano matroid, defined as in the following picture 1, is realizable over any field of characteristic* 2*, and not over any other characteristics.*

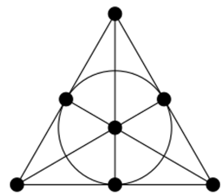

Figure 1: fano matroid

Example 2.3. *The non-Pappus matroid, defined as in the following picture 2, is not realizable over any field.*

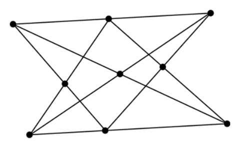

Figure 2: non-Pappus matroid

2.2 Matroid operations

Given a matroid M and an element e in the ground set E, we can form two new matroids $M \setminus e$ and M/e, the deletion and contraction of e. They corresponds to the deletion and contraction of an edge in a graph.

Definition 2.10 (deletion matroid and contraction matroid)**.** *Let M be a matroid with ground set E. For any $e \in E$, we define the **deletion matroid** $M \setminus e$ to be the matroid on $E \setminus \{e\}$ with rank functions $\mathrm{rk}_{M\setminus e}(S) = \mathrm{rk}_M(S)$ for any $S \subset E$. We define the **contraction matroid** M/e to be the matroid on $E \setminus \{e\}$ with rank function* $\mathrm{rk}_{M/e}(S) = \mathrm{rk}(S \cup \{e\}) - r(e)$.

Remark 2.5. *Equivalently, the deletion and contraction can be defined using independent sets:*

$$\mathcal{I}_{M\setminus e} = 2^{E\setminus\{e\}} \cap \mathcal{I}_M$$

and

$$\mathcal{I}_{M/e} = \{I \subset E \setminus e \mid I \cup \{e\} \in \mathcal{I}_M\}.$$

Remark 2.6. *If e is a loop, then $M \setminus e = M/e$, whose flats are naturally bijective the the flats of M. If e is not a loop, then $\mathrm{rk}(M/e) = \mathrm{rk}(M) - 1$. On the other hand, if e appears in every basis (in this case, e is called a **coloop**), then $\mathrm{rk}(M \setminus e) = \mathrm{rk}(M) - 1$; otherwise,* $\mathrm{rk}(M \setminus e) = \mathrm{rk}(M)$.

Definition 2.11. *Given a flat F of a matroid M, we define the matroid M^F to be the matroid deleting every element of $E \setminus F$ from M. We also define the matroid M_F to be the matroid contracting every element of F from M.*

Remark 2.7. *If M is a simple matroid, then M^F is also simple for any flat F. However, M_F may fail to be simple. If M is a loopless matroid, then M^F and M_F are loopless for any flat F.*

Definition 2.12 (direct sum)**.** *Another way to construct new matroid is taking direct sum. Let M_1 and M_2 be matroids with ground sets E_1 and E_2 respectively. Then their **direct sum** $M_1 \oplus M_2$ is defined to be the matroid with ground set $E_1 \sqcup E_2$ and*

$$\mathcal{I}_{M_1 \oplus M_2} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\}.$$

Remark 2.8. *Equivalently, the direct sum matroid can be defined using basis*

$$\mathcal{B}_{M_1 \oplus M_2} = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}_{M_1},\, B_2 \in \mathcal{B}_{M_2}\},$$

or rank function

$$\mathrm{rk}_{M_1 \oplus M_2}(S_1 \cup S_2) = \mathrm{rk}_{M_1}(S_1) + \mathrm{rk}_{M_2}(S_2),$$

or flats

$$\mathcal{F}_{M_1 \oplus M_2} = \{F_1 \cup F_2 \mid F_1 \in \mathcal{F}_{M_1}, F_2 \in \mathcal{F}_{M_2}\}.$$

2.3 Characteristic polynomials

Definition 2.13 (characteristic polynomial)**.** *Let M be a matroid with ground set E. Its characteristic polynomial is defined as*

$$\chi_M(t) = \sum_{S \subset E} (-1)^{|S|} t^{r(M) - r(S)}.$$

Proposition 2.2. *The characteristic polynomial $\chi_M(t)$ satisfies the following properties:*

1. ***(loop property)*** *If M has a loop, then $\chi_M(t) = 0$;*
2. ***(normalization)*** *The characteristic polynomial of the uniform matroid $U_{1,1}$ satisfies*
$$\chi_{U_{1,1}}(t) = t - 1;$$
3. ***(direct sum)*** *If $M = M_1 \oplus M_2$, then*
$$\chi_{M_1 \oplus M_2}(t) = \chi_{M_1}(t) \cdot \chi_{M_2}(t);$$

*4. (**deletion/contraction**) If e is not a coloop of M, then*

$$\chi_M(t) = \chi_{M\setminus e}(t) - \chi_{M/e}(t).$$

Moreover, the characteristic polynomial is the unique way to associate each matroid a polynomial such that all the above properties are satisfied.

Proposition 2.3. *Let G be a graph, and let M be the associated matroid. Then,*

$$P_G(t) = t^l \cdot \chi_M(t)$$

where l is the number of connected components of G.

2.4 Hyperplane arrangement

As we have discussed, a realizable matroid corresponds to a vector configuration. Dually, a (loopless) realizable matroid also corresponds to a hyperplane arrangement.

Definition 2.14. *Fixing a d-dimensional vector space V, a **central hyperplane arrangement** is a collection $\mathcal{A} = \{H_1, \ldots, H_n\}$ of $(d-1)$-dimensional linear subspaces (also called hyperplanes). Here we allow two hyperplanes to be the same. The hyperplane arrangement $\mathcal{A}$ is called **essential**, if the intersection of all the hyperplanes H_i is equal to zero. Given a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in V, we define the hyperplane arrangement complement to be the open subvariety $U = V \setminus (H_1 \cup \cdots \cup H_n)$ of V.*

Proposition 2.4. *Given such a central hyperplane arrangement $\mathcal{A}$, it defines a matroid M on $E = \{1, \ldots, n\}$ whose rank function is given by*

$$r(S) = d - \dim \bigcap_{i \in S} H_i.$$

In the realizable case, the characteristic polynomial of a matroid is closely related to the geometry of the corresponding hyperplane arrangement complement.

Proposition 2.5. *Let M be the loopless matroid associated to an essential hyperplane arrangement $\mathcal{A}$ in a K-vector space V, and let U be the hyperplane arrangement complement. If $\chi_M(t) = t^d + a_1 t^{d-1} + \cdots + a_d$, then in $K_0(\mathrm{Var}_K)$, the Grothendieck ring of algebraic varieties over K,*

$$[U] = \mathbb{L}^d + a_1 \mathbb{L}^{d-1} + \cdots + a_d,$$

where $\mathbb{L} = [\mathbb{A}^1]$.

As consequences of the above proposition, we have the following.

Corollary 2.1. *Let $\mathcal{A}$ be an essential hyperplane arrangement in a vector space V over a finite field $\mathbb{F}_q$ with q elements, and let U be the complement. Denote the associated matroid to be M. Then the number of $\mathbb{F}_{q^r}$ points on U is qual to $\chi_M(q^r)$.*

Corollary 2.2. *Let $\mathcal{A}$ be an essential hyperplane arrangement in a complex vector space V, and let U be the complement. Denote the associated matroid to be M. Then,*

$$\chi_M(t) = \sum_{0\le k\le d} (-1)^k \dim_{\mathbb{Q}} \mathrm{H}^k(U,\mathbb{Q}) t^{d-k}$$

$$= \sum_{0\le k\le d} (-1)^{d-k} \dim_{\mathbb{Q}} \mathrm{H}_c^{d+k}(U,\mathbb{Q}) t^k.$$

Proof. **Sketch of proof.** The two summations are equal to each other by Poincaré duality. So it is enough to show $\chi_M(t)$ is equal to the second summation.

To relate the Betti numbers and the class in the Grothendieck ring, we need a fact that the mixed Hodge structure on $\mathrm{H}^k(U,\mathbb{Q})$ is of (k,k)-type. In fact, when $k = 1$, this follows from the fact that U admits a good compactification $U \subset \mathbb{P}^{d-1}$ with $\mathrm{H}^1(\mathbb{P}^{d-1},\mathbb{Q}) = 0$. In general, we know that the cohomology ring $\mathrm{H}^\bullet(U,\mathbb{Q})$ is generated in degree one (e.g., by the theorem of Orlik-Solomon). Thus, the mixed Hodge structure on $\mathrm{H}^k(U,\mathbb{Q})$ is of (k,k)-type. By Poincaré duality, the Hodge structure on $\mathrm{H}_c^k(U,\mathbb{Q})$ is of $(k-d,k-d)$-type. The class $[U] \in K_0(\mathrm{Var}_K)$ determines the (compactly supported) Hodge-Deligne polynomial of U:

$$E(U) = \sum_k \sum_{p,q} (-1)^k h^{p,q}\left(\mathrm{H}_c^k(U,\mathbb{Q})\right) x^p y^q$$

where $h^{p,q}\left(\mathrm{H}_c^k(U,\mathbb{Q})\right)$ denotes the dimension of the (p,q)-component of the mixed Hodge structure on $\mathrm{H}_c^k(U,\mathbb{Q})$. Since $\mathrm{H}_c^k(U,\mathbb{Q})$ is of $(k-d,k-d)$-type, we know that

$$E(U) = \sum_{d\le k\le 2d} (-1)^k \dim \mathrm{H}_c^k(U,\mathbb{Q})(xy)^{k-d}$$

$$= \sum_{0\le k\le d} (-1)^k \dim \mathrm{H}_c^{2d-k}(U,\mathbb{Q})(xy)^{d-k}.$$

On the other hand, since $E(\mathbb{C}^k) = (xy)^k$, by Proposition 2.5,

$$E(U) = \chi_M(xy).$$

Combining the above two equalities and substitute k by $d-k$, we have the desired equality

$$\chi_M(t) = \sum_{0\le k\le d} (-1)^{d-k} \dim \mathrm{H}_c^{d+k}(U,\mathbb{Q}) t^k.$$

□

Remark 2.9. *The above two corollaries imply that the Betti numbers of a hyperplane arrangement complement (over* $\mathbb{C}$*) and the number of* $\mathbb{F}_q^r$ *points on the complement (over a finite field) are combinatorial invariants. A deeper theorem of Orlik-Solomon says that the cohomology ring of a hyperplane arrangement complement (over* $\mathbb{C}$*) is also a combinatorial invariant. In this note, we will not get into details along this direction.*

Remark 2.10. *In [3], Huh proved the log-concavity of the coefficients of the characteristic polynomial for matroids realizable over a field of characteristic* 0*. He used the above corollary (more precisely the projective analog) and a theorem of Dimca-Papadima to realize the coefficients of the characteristic polynomial as intersections numbers on a projective variety. In the later paper of Huh and Katz [4], they used combinatorial arguments to show the coefficients of the (reduced) characteristic polynomial are the desired intersection numbers.*

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