Notes on GCMC

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Abstract

This article consists of some notes taken by the author while studying **Gaussian completely monotone conjecture**.

References: [1], [2], [3], [4].

References

[1] Guangyue Han and Jian Song. "Extensions of the I-MMSE relationship to Gaussian channels with feedback and memory". In: *IEEE Transactions on Information Theory* 62.10 (2016), pp. 5422–5445.

[2] Max Costa. "A new entropy power inequality". In: *IEEE Transactions on Information Theory* 31.6 (1985), pp. 751–760.

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[4] Fan Cheng and Yanlin Geng. "Higher order derivatives in Costa's entropy power inequality". In: *IEEE Transactions on Information Theory* 61.11 (2015), pp. 5892–5905.

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2.1 Proof given by Prof. Han

The following de Bruijn's identity is a fundamental relationship between the differential entropy and the Fisher information. Based on the proposed approach, we will give a new proof of this classical result.

Theorem 2.1. *Let X be any random variable with a finite variance and let Z be an independent standard normally distributed random variable. Then, for any $t > 0$,*

$$\frac{d}{dt}H(X+\sqrt{t}Z) = \frac{1}{2}J(X+\sqrt{t}Z), \tag{2.1}$$

where $J(\cdot)$ denotes the Fisher information.

Proof. First of all, define

$$Y = X + \sqrt{t}Z,$$

whose density function can be computed as

$$f_Y(y) = \int_{\mathbb{R}} f_X(x) f_{Y|X}(y|x)dx = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}dx.$$

Here, to prevent possible confusion, we remark that Y defined as above should be regarded as "local" to this proof, as the same notation is used to denote the output of Gaussian channels elsewhere in this paper. Immediately, we have

$$f_Y(Y) = f_Y(X+\sqrt{t}Z) = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(X+\sqrt{t}Z-x)^2/(2t)}dx.$$

Now, taking the derivative with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt}f_Y(Y) &= \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(X+\sqrt{t}Z-x)^2/(2t)} \left(\frac{(X-x)(X+\sqrt{t}Z-x)}{2t^2} - \frac{1}{2t} \right) dx \\
&= \int_{\mathbb{R}} \left(\frac{(X-x)(X+\sqrt{t}Z-x)}{2t^2} - \frac{1}{2t} \right) f_{Y|X}(Y|x) f_X(x)dx \\
&= f_Y(Y) \int_{\mathbb{R}} \left(\frac{(X-x)(Y-x)}{2t^2} - \frac{1}{2t} \right) f_{X|Y}(x|Y)dx,
\end{aligned}$$

It then follows that

$$
\begin{aligned}
\frac{d}{dt}H(Y) &= -\frac{d}{dt}\mathbb{E}[\log f_Y(Y)] \\
&= -\mathbb{E}\left[\frac{1}{f_Y(Y)}\frac{d}{dt}f_Y(Y)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}}\left(-\frac{(X-x)(Y-x)}{2t^2}+\frac{1}{2t}\right)f_{X|Y}(x|Y)dx\right] \\
&= \mathbb{E}[-XY+(X+Y)\mathbb{E}[X|Y]-\mathbb{E}[X^2|Y]]\frac{1}{2t^2}+\frac{1}{2t} \\
&= \frac{-\mathbb{E}[X^2]+\mathbb{E}[\mathbb{E}^2[X|Y]]}{2t^2}+\frac{1}{2t} \\
&= \frac{-\mathbb{E}[X^2]+\mathbb{E}[\mathbb{E}^2[X|Y]]+\mathbb{E}[(X-Y)^2]}{2t^2} \\
&= \frac{\mathbb{E}[\mathbb{E}^2[X|Y]]+\mathbb{E}[Y^2]-2\mathbb{E}[XY]}{2t^2},
\end{aligned}
$$

On the other hand, similarly as above, we derive

$$
f'_Y(Y) = \int_{\mathbb{R}}\frac{f_X(x)}{\sqrt{2\pi t}}e^{-(Y-x)^2/(2t)}\frac{x-Y}{t}dx = f_Y(Y)\int_{\mathbb{R}}\frac{x-Y}{t}f_{X|Y}(x|Y)dx,
$$

where $f'_Y(\cdot)$ means the derivative of the function of $f_Y(\cdot)$ with respect to its parameter. It then follows that

$$
\begin{aligned}
J(Y) = \mathbb{E}&\left[\left(\frac{f'_Y(Y)}{f_Y(Y)}\right)^2\right] \\
&= \frac{\mathbb{E}[\mathbb{E}^2[X|Y]+Y^2-2\mathbb{E}[X|Y]Y]}{t^2} \\
&= \frac{\mathbb{E}[\mathbb{E}^2[X|Y]]+\mathbb{E}[Y^2]-2\mathbb{E}[XY]}{t^2},
\end{aligned}
$$

which immediately implies the desired (2.1). □

2.2 Proof given by Prof. Cheng

3 Heat equation proof of the Hodge theorem

3.1 Introduction

The most basic PDE is Laplace's equation. For real-valued functions $f : U \to \mathbb{R}$ defined on some domain $U \subset \mathbb{R}^n$, we want to find solutions to

$$
\Delta f = 0 \tag{3.1}
$$

where Δ is the **Laplace operator** or **Laplacian**, (our convention is to include a minus sign)

$$\begin{aligned}\Delta &= -\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)\\ &= -\operatorname{div}\circ\operatorname{grad}\\ &= -\operatorname{Tr}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\end{aligned}$$

Solutions to (3.1) are called **harmonic functions**.

In looking for harmonic functions, one method is to look at solving the **heat equation**. Here we look for functions $f : [0,\infty)\times U \to \mathbb{R}$ which solve

$$\left(\frac{\partial}{\partial t}+\Delta\right) f(t,x) = 0 \tag{3.2}$$

with some initial condition $f(0,x) = f_0(x)$ for a fixed function $f_0 : U \to \mathbb{R}$.

Both equations (3.1) and (3.2) come from physics. Equation (3.2) is meant to describe the way the temperature of a physical object (with shape U) redistributes itself over time. Here the value $f(t,x)$ would be the temperature at the point x at time t. We say that f follows the heat flow, and (3.2) is also sometimes called the heat flow equation.

Why would this equation help with Laplace's equation? Well it turns out to be natural to hope that, if we can solve this equation for all time t, at $t=\infty$ we reach some "steady state" $\frac{\partial}{\partial t} f = 0$ (i.e. the heat has distributed evenly and the temperature has settled into equilibrium), in which case we have $\Delta f = 0$. In other words we hope that solutions to the heat equation "flow" to harmonic functions in the limit.

What we are interested in is solving Laplace's equation for differential forms $\omega \in \Omega^\bullet(M)$ on a compact Riemannian manifold (M,g).

$$\Delta\omega = 0 \tag{3.3}$$

We will need the Riemannian structure g to define the new Laplacian $\Delta = \Delta_g$ on differential forms. Solutions to (3.3) are called <u>harmonic forms</u> (or <u>harmonic k-forms</u> if $\omega \in \Omega^k$). Since Δ is linear, the solution space will be a vector space. We denote

$$\mathcal{H}^k := \{\text{harmonic } k\text{-forms}\}$$

It will turn out that harmonic forms reveal topological information about M. In particular, we will see that the space of harmonic k-forms is isomorphic to the kth deRham cohomology. This is the Hodge theorem.

Theorem 3.1 (Hodge)**.** *For each k,*

$$\mathcal{H}^k \cong H^k_{dR}. \tag{3.4}$$

Furthermore, each cohomology class has a unique harmonic representative.

What's interesting here is that we've equated something geometric (the Laplacian Δ and therefore also $\mathcal{H}^k$ depend on the choice of metric g) to the cohomology which only depends on the topology of M. What's even more interesting is revealed in the very constructive heat equation proof below: the isomorphism can be obtained by just following the heat flow to $t = \infty$.

3.2 Example: Heat Equation in one dimension

On $M = S^1 = [0, 2\pi]/\{0, 2\pi\}$, the heat equation is

$$(\partial_t - \partial_\theta^2) f(t, \theta) = 0.$$

where θ is the coordinate on S^1. We can solve this explicitly using Fourier series.

Let $f_0(\theta)$ be the initial condition. Every function on S^1 has a Fourier expansion,

$$f_0(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$$

If f_0 has a solution f to the heat flow (i.e. a function $f(t, \theta)$ with $f(0, \theta) = f_0(\theta)$), it must also have a (time-dependent) Fourier expansion

$$f(t, \theta) = \sum_n a_n(t) e^{in\theta}$$

which satisfies

$$\begin{aligned} 0 &= (\partial_t - \partial_\theta^2) \sum_n a_n(t) e^{in\theta} \\ &= \sum_n \left(a_n'(t) + n^2 a_n(t) \right) e^{in\theta} \end{aligned}$$

We are left with the differential equation

$$a_n'(t) = -n^2 a_n(t)$$

which has solutions

$$a_n(t) = a_n e^{-n^2 t}$$

where the a_n's are from the initial condition f_0. So

$$f(t, \theta) = a_n e^{in\theta} e^{-n^2 t}. \tag{3.5}$$

Now we ask, what happens as $t \to \infty$? Because of the exponential, every term in the sum in (3.5) dies out except for $n = 0$. Hence

$$\lim_{t\to\infty} f(t, \theta) = a_0$$

Notice that this is the average value of f_0.

What happened is that all the other Fourier modes (which have an average value of 0) all distributed themselves evenly in the circle. So e.g. a sine wave, representing the temperature on the circle, eventually shrinks to zero as heat flows from the peak to the dip.

In contrast with the case of the line $\mathbb{R}$, the heat has nowhere to go, so we are left with the average total temperature we started with.

3.3 Differential forms and the Laplacian

In this section we quickly give the set up to explain what the two sides of (3.4) are.

Let M be a closed (compact with no boundary) smooth manifold.

Recall that we have the exterior derivative operator d on the space of differential forms

$$\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^n$$

which forms a complex ($\mathrm{d}^2 = 0$). This gives us

Definition 3.1 (de Rham cohomology). *The* ***de Rham cohomology*** *of* M

$$H^k_{dR} := \frac{\ker d \cap \Omega^k}{\operatorname{im} d}$$

DeRham's theorem says that the deRham cohomology is isomorphic to the singular cohomology, so H_{dR} is topological. It doesn't even depend on the smooth structure of M.

We continue on to define the Laplacian

Definition 3.2 (Riemannian metric). *A* ***Riemannian metric*** g *on* M *defines an inner product on each tangent space* T_xM *which in turn induces one on each exterior cotangent space* $\wedge^k T_x^* M$*. It also defines a canonical* <u>*Riemannian volume form*</u> $\mathrm{dvol}_g \in \Omega^n$ *against which we can integrate smooth functions. This data defines an*

L^2 *inner product on the space* Ω^k *of* k*-forms: For* $\omega, \tau \in \Omega^k$,

$$\langle \omega, \tau \rangle_{L^2} := \int_{x \in M} \langle \omega(x), \tau(x) \rangle_g \ \mathrm{dvol}_g$$

All this is just to allow us to define

Definition 3.3 (codifferential operator)**.** *The* ***codifferential operator*** d^*,

$$d^* : \Omega^{k+1} \to \Omega^k,$$

is the L^2*-adjoint of* d*. That is,* d^* *is defined by: For* $\omega \in \Omega^k$ *and* $\tau \in \Omega^{k+1}$

$$\langle d\omega, \tau \rangle_{L^2} = \langle \omega, d^* \tau \rangle_{L^2}$$

Notice that, just like d, we have $(\mathrm{d}^*)^2 = 0$, because

$$\langle (\mathrm{d}^*)^2 \omega, \tau \rangle_{L^2} = \langle \omega, \mathrm{d}^2 \tau \rangle_{L^2} = 0$$

for every ω, τ. Thus we have a **double complex**

$$\Omega^0 \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1 \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^2 \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots \underset{\mathrm{d}^*}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^n$$

Definition 3.4 (Hodge-deRham Laplacian)**.** *For each* k*, we define the* ***Hodge-deRham Laplacian*** *on differential* k*-forms to be the operator*

$$\Delta = dd^* + d^* d : \Omega^k \to \Omega^k$$

As justification to why we call this a Laplacian, we point out that

1. This Laplacian generalizes the classical one. For smooth functions $f \in C^\infty = \Omega^0$ on $\mathbb{R}^n$ with its standard Euclidean metric, the Hodge Laplacian reduces to

   $$\Delta = \mathrm{d}^* \mathrm{d}$$

   because d^* on Ω^0 is 0. If we write the operators in coordinates, we would see that d is the gradient, and d^* is minus the divergence. Hence

   $$\Delta = -\mathrm{div} \circ \mathrm{grad}$$

   as before.

2. If we were to write the general Hodge Laplacian on k-forms in local coordinates

on M, we would get

$$\Delta\omega = \Delta \sum_i \omega_i dx^i = \sum_i \left(\sum_j \frac{\partial^2 \omega_i}{\partial x_j^2} \right) dx^i + (\text{lower order terms})$$

i.e. the highest order derivative terms look like the standard Laplacian on the component functions ω_i. For PDE's this highest order term is the most important.

Definition 3.5 (space of harmonic k-forms)**.** *We define the **space of harmonic k-forms** $\mathcal{H}^k$ as the kernel of Δ in Ω^k.*

$$\mathcal{H}^k = \{\omega \in \Omega^k \ : \ \Delta\omega = 0\}$$

Remark 3.1. *Suppose ω is a harmonic k-form. Then we have*

$$\begin{aligned} 0 &= \langle \Delta\omega, \omega \rangle_{L^2} = \langle d^* d\omega, \omega \rangle + \langle dd^* \omega, \omega \rangle \\ &= \langle d\omega, d\omega \rangle + \langle d^* \omega, d^* \omega \rangle \\ &= \| d^* \omega \|_{L^2}^2 + \| d\omega \|_{L^2}^2 \end{aligned}$$

*which implies that $d\omega = d^*\omega = 0$, i.e. that ω is **closed** ($d\omega = 0$) and **co-closed** ($d^*\omega = 0$).*

Conversely, just by definition of $\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$, if ω is closed and co-closed then it is harmonic.

This proves

$$\Delta\omega = 0 \iff \begin{cases} \mathrm{d}\omega = 0 \\ \mathrm{d}^*\omega = 0 \end{cases} \tag{3.6}$$

We can finally write down the heat equation for differential forms on M.

$$\left(\frac{\partial}{\partial t} + \Delta \right) \omega = 0$$

3.4 Proof of the Hodge Theorem using the heat flow

We first ask: Can we solve the heat equation? In short, yes. And we will not go into how.

A more precise statement is

Theorem 3.2. *Given any smooth (or even L^2) differential form $\omega_0 \in \Omega^k$, there exists a unique solution to the heat equation, i.e. a unique smooth map*

$$\omega : [0, \infty) \to \Omega^k \quad t \mapsto \omega_t$$

such that

$$\left(\frac{\partial}{\partial t} + \Delta\right)\omega_t = 0$$

with initial condition $0 \mapsto \omega_0$.

Given this fact, we define the following.

Definition 3.6 (heat equation solver operator)**.** *For* $t \in [0, \infty)$, *let* H_t *be the* ***time-t-heat-equation-solver operator****, which takes a form* ω_0 *to its solution under the heat flow at time* t. *So* $H_t\omega_0 = \omega_t$.

In other words, for each t,

$$H_t : \Omega^k \to \Omega^k$$

and for any $\omega \in \Omega^k(M)$ *we have*

$$\left(\frac{\partial}{\partial t} + \Delta\right) H_t\omega = 0$$

$$H_0\omega = \omega$$

Remark 3.2. *The operator* H_t *can also be denoted by* $e^{-t\Delta}$. *This is inspired by the fact that*

$$\partial_t\left(e^{-t\Delta}\omega_0\right) = -\Delta\left(e^{-t\Delta}\omega_0\right).$$

looks like it makes sense. This is also closer to being true with rigorous analysis.

Remark 3.3 (Semi-group property)**.** *The operator* H_t *says "flow this differential form for a time* t*". By uniqueness of the solution, we have for example,* $H_{t+s} = H_t \circ H_s$ *(this looks nice in exponential notation), since these both say: flow for a time* s*, then for a time* t*. We also have some necessary properties which are not so obvious*

Lemma 3.1. *The operator* H_t *commutes with both* Δ *and* d.

We will need one last thing before proving Theorem 3.1.

Lemma 3.2. *In Example (3.5), it was quite useful to use the Fourier modes. This is because they form simultaneous eigen-decompositions of the operators* Δ *and* H_t:

$$\Delta e^{in\theta} = \lambda_n e^{in\theta}$$

$$H_t e^{in\theta} = e^{-\lambda_n t} e^{in\theta}$$

with $\lambda_n = n^2$, *for* $n = 0, 1, 2, \ldots$.

For closed Riemannian manifolds M, *we also have similar eigen-decompositions for* Δ *and* H_t*: There exist* $\{\lambda_n \in \mathbb{R}\}$ *and* L^2*-orthogonal* $\{\zeta_n \in \Omega^k\}$ *for* $n = 0, 1, 2, \ldots$, *with*

$$\Delta\zeta_n = \lambda_n\zeta_n$$

$$H_t\zeta_n = e^{-\lambda_n t}\zeta_n$$

such that every k-form has a unique decomposition into a combination of ζ_n's. Furthermore each eigenvalue satisfies $\lambda_n \geq 0$ and has finite multiplicity.

Note that $\lambda = 0$ corresponds to the kernel of Δ. This spectral decomposition proves that $\mathcal{H}^k$ is finite dimensional.

We finally have all the ingredients for the proof of the Hodge isomorphism theorem.

Proof. Define the linear map I which sends a harmonic form to its cohomology class.

$$I : \mathcal{H}^k \xrightarrow{\sim} H^k_{\mathrm{dR}}$$

$$\omega \mapsto [\omega]$$

This makes sense because $\omega \in \mathcal{H}^k$ is closed by (3.6).

The map I is **injective**:
If $[\omega] = 0$, then $\omega = \mathrm{d}\tau$ for some τ. Since ω is co-closed by (3.6),

$$0 = \langle \mathrm{d}^*\omega, \tau\rangle_{L^2} = \langle \mathrm{d}^*\mathrm{d}\tau, \tau\rangle_{L^2} = \|\mathrm{d}\tau\|^2_{L^2} \tag{3.7}$$

which proves $\omega = \mathrm{d}\tau = 0$. Injectivity says that the harmonic representative of a cohomology class is unique.

Finally, I is **surjective**:
Let ω be a closed k-form, a representative of the class $[\omega]$. Then $H_t\omega$ is its solution to the heat flow.

We already have $\mathrm{d} \circ H_t = H_t \circ \mathrm{d}$, so H_t takes closed forms to closed forms, but more explicitly:

$$\begin{aligned}
H_t\omega - \omega = H_t\omega - H_0\omega &= \int_0^t \partial_t H_t\omega \\
&= -\int_0^t \Delta H_t\omega \\
&= -\int_0^t H_t\Delta\omega \qquad \text{since } H_t \text{ commutes with } \Delta \\
&= -\int_0^t H_t(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega \\
&= -\int_0^t H_t\mathrm{d}\mathrm{d}^*\omega \qquad \text{since } \omega \text{ is closed} \\
&= -\int_0^t \mathrm{d}H_t\mathrm{d}^*\omega \qquad \text{since } H_t \text{ commutes with d} \\
&= \mathrm{d}\left[-\int_0^t H_t\mathrm{d}^*\omega\right]
\end{aligned}$$

is an exact form. Thus the heat flow preserves the cohomology class of ω (the flow is *cohomologous*): $[H_t\omega] = [\omega]$ for all time t.

What happens as $t \to \infty$? Using the spectral decomposition of Δ, we have

$$\omega = \sum_n a_n \zeta_n$$

so

$$H_t\omega = \sum_n a_n H_t \zeta_n = \sum_n a_n e^{-\lambda_n t} \zeta_n.$$

Thus

$$\omega_\infty := \lim_{t\to\infty} H_t\omega = \sum_n a_n \zeta_n \left(\lim_{t\to\infty} e^{-\lambda_n t} \right) = \sum_{n=0}^{N} a_n \zeta_n$$

where $\zeta_n, n = 0, 1, \ldots, N$ span the kernel of Δ ($\lambda = 0$), i.e. harmonic.

So the long time limit ω_∞ of $H_t\omega$ is harmonic! And since the flow keeps the form within a single cohomology class, ω_∞ is a harmonic representative of $[\omega]$. $\square$

3.5 Proof using Hodge Decomposition

There is another proof of the Hodge theorem, using the following decomposition theorem, one proof of which comes from applying elliptic theory to the operator $D := d + d^* : \Omega^\bullet \to \Omega^\bullet$. Note that $D^2 = \Delta$ and by (3.6), $\ker D = \ker \Delta$.

Theorem 3.3 (Hodge Decomposition)**.** *The space of differential forms decomposes L^2-orthogonally as*

$$\Omega = \mathcal{H} \oplus d\Omega \oplus d^*\Omega$$

In particular, since $\Omega = \bigoplus_k \Omega^k$,

$$\Omega^k = \mathcal{H}^k \oplus d\Omega^{k-1} \oplus d^*\Omega^{k+1}$$

Proof. **Proof of 3.1.** Given a closed form $\omega \in \Omega^k$, ω decomposes uniquely as

$$\omega = h + \mathrm{d}b + \mathrm{d}^*c$$

where $h \in \mathcal{H}^k$.

Since ω is closed,

$$0 = \mathrm{d}\omega = \mathrm{d}h + \mathrm{d}^2 b + \mathrm{d}\mathrm{d}^* c = \mathrm{d}\mathrm{d}^* c.$$

But $\mathrm{d}\mathrm{d}^* c = 0$ implies that

$$0 = \langle \mathrm{d}\mathrm{d}^* c, c \rangle_{L^2} = \|\mathrm{d}^* c\|_{L^2}$$

so $\mathrm{d}^* c = 0$.

Thus we actually have

$$\omega = h + \mathrm{d}b$$

That is, h is harmonic and represents the same cohomology class as ω. We may define the map $\omega \mapsto h$. $\square$

3.6 Spectral Decomposition

We describe how we arrive at the simultaneous spectral decompositions of Δ and H_t in Lemma 3.2. We let L^2 denote the space $L^2(\Omega^k)$, the completion of Ω^k under the L^2 norm. Then Δ and H_t can both be considered as operators $L^2 \to L^2$ (Δ is an **unbounded** operator on L^2).

Recall that in Example (3.5), we said the heat flow **smooths** functions. This turns out to be generally true.

The operator H_t is defined for all L^2 forms using a smooth heat kernel:

$$(H_t\omega)(x) = \int_{y\in M} \langle e_t(x,y), \omega(y)\rangle \mathrm{dvol}_g$$

which satisfies $e_t(x,y) = e_t(y,x)$. From this symmetry property, one can show that the operator H_t is self-adjoint.

It is also true that, for all $t > 0$, the form $H_t\omega$ lies in $C^\infty(\Omega^k) = \Omega^k$, (derivatives ∂_x of $H_t\omega(x)$ apply directly to $e_t(x,y)$ within the integral). In particular, $H_t\omega \in L^{1,2}$ which embeds **compactly** into L^2 by Sobolev embedding.

Thus the map $H_t : L^2 \to L^2$ is a compact self-adjoint operator. For such operators we have

Theorem 3.4 (Spectral Theorem for Compact Self-Adjoint Operators)**.** *Let $T : H \to H$ be a compact self-adjoint operator on an infinite dimensional Hilbert space. Then H decomposes into an orthogonal countable sum of eigenspaces of T, each of finite dimension. Each eigenvalue γ_n is real with finite multiplicity, and we have $\gamma_n \to 0$.*

$$H = \bigoplus_{n=0}^{\infty} \langle v_n \rangle$$

$$T v_n = \gamma_n v_n$$

Note that in our case we have a *family* of compact self-adjoint operators $\{H_t, t > 0\}$. This gives us a *family* of decompositions of L^2 into eigenspaces.

$$L^2 = \bigoplus_{n=0}^{\infty} \langle \zeta_n(t) \rangle$$

$$H_t \zeta_n(t) = \gamma_n(t) \zeta_n(t).$$

with $\gamma_n(t) \in \mathbb{R}$.

However, since $H_t H_s = H_{t+s} = H_s H_t$, the operators $\{H_t\}$ are simultaneously diagonalizable. That is, the eigenvectors ζ_n and the decomposition of L^2 don't depend on t.

$$L^2 = \bigoplus_{n=0}^{\infty} \langle \zeta_n \rangle$$

$$H_t \zeta_n = \gamma_n(t) \zeta_n.$$

Lemma 3.3. *The eigenvalues $\gamma_n(t)$ of H_t are strictly positive for $t > 0$.*

Proof. 1. $\gamma_n(t) \geq 0$:

$$\gamma_n(t) = \langle H_t \zeta_n, \zeta_n \rangle_{L^2} = \langle H_{t/2} \zeta_n, H_{t/2} \zeta_n \rangle_{L^2} = \|H_{t/2} \zeta_n\|_{L^2}^2 \geq 0$$

2. $\gamma_n(t) \neq 0$: Suppose $H_t \zeta = 0$ for some t and ζ. We will show that ζ must be 0. The same trick as before:

$$0 = \langle H_t \zeta, \zeta \rangle_{L^2} = \langle H_{t/2} \zeta, H_{t/2} \zeta \rangle_{L^2} = \|H_{t/2} \zeta\|_{L^2}^2$$

implies that $H_{t/2}\zeta = 0$ also. We can keep doing this to show that $H_{t/2^m} = 0$ for all $m \in \mathbb{N}$. Then

$$0 = \lim_{m \to \infty} H_{t/2^m} \zeta = \lim_{t \to 0} H_t \zeta = \zeta$$

Thus H_t is injective for each $t > 0$. (This last limit follows because the rigorous meaning of $H_0 \omega = \omega$ in the definition of H_t is that $\lim_{t \to 0} H_t \omega = \omega$.)

□

The spectral theorem allowed us to diagonalize H_t. It is further true that the ζ_n's diagonalize Δ: Because $H_t \zeta_n = \gamma_n(t) \zeta_n$ solves the heat equation,

$$0 = (\partial_t + \Delta) \gamma_n(t) \, \zeta_n = \gamma_n'(t) \, \zeta_n + \gamma_n(t) \, \Delta \zeta_n$$

$$\Delta \zeta_n = -\frac{\gamma_n'(t)}{\gamma_n(t)} \zeta_n$$

(We can divide by $\gamma_n(t)$ because they are nonzero.) Notice that $\Delta \zeta_n$ doesn't depend on t, which implies that the right side of the equation doesn't either. That is,

$$\lambda_n := -\frac{\gamma_n'(t)}{\gamma_n(t)} = \text{const.}$$

$$\gamma_n'(t) = -\lambda_n \gamma_n(t)$$

$$\gamma_n(t) = C_n e^{-\lambda_n t}$$

But $H_0 = \mathrm{id}$ means that for each n, we have $\gamma_n(t) \xrightarrow{t \to 0} 1$ which forces $C_n = 1$.

$$\gamma_n(t) = e^{-\lambda_n t}$$

In summary we have

$$\Delta \zeta_n = \lambda_n \zeta_n$$

$$H_t \zeta_n = e^{-\lambda_n t} \zeta_n$$

We note that $\gamma_n(t) \xrightarrow{n \to \infty} 0$ implies that $\lambda_n \xrightarrow{n \to \infty} +\infty$.
We also have from the definition of Δ:

$$\lambda_n = \langle \Delta \zeta_n, \zeta_n \rangle = \|d^* \zeta_n\|^2 + \|\mathrm{d} \zeta_n\|^2 \geq 0$$