Complex geometry and Hodge theory

Galobel Wang
The University of Hong Kong

September 2025

Abstract

This article consists of some notes taken by the author while studying **Complex geometry and Hodge theory**.

References: [1], [2], [3], [4].

References

[1] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* John Wiley & Sons, 2014.

[2] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II: Volume 2.* Vol. 77. Cambridge University Press, 2003.

[3] Daniel Huybrechts. *Complex geometry: an introduction.* Springer, 2005.

[4] Eduardo Cattani et al. *Hodge theory.* Vol. 49. Princeton University Press, 2014.

Contents

1 Introduction to Complex algebraic geometry

Let $f_i(x_1, \cdots, x_n),\ i \in \{1, \cdots, k\}$ be polynomials with coefficients in $\mathbb{R}$ and $\mathbb{C}$. An **affine algebraic variety** is the common zero set of

$$X = X(f_1, \cdots, f_n) = \{x : f_i(x) = 0,\ \forall i\}.$$

We incorporate the field into the notation

$$X(\mathbb{C}) = \{x \in \mathbb{C}^n : f_i(x) = 0,\ \forall i\}$$

$$X(\mathbb{R}) = \{x \in \mathbb{R}^n : f_i(x) = 0,\ \forall i\} \quad (\text{when } f_i \in \mathbb{R}[x]).$$

These can be thought of naturally as topological spaces with the topology they inherit from $\mathbb{C}^n$ or $\mathbb{R}^n$. (Alternatively, one can see the Zariski topology induced by declaring that zero sets of polynomials are closed.)

Remark 1.1. *$X(\mathbb{C})$ is essentially never compact.*

To remedy this, we shift our attention to projective space.

Definition 1.1 (complex projective space)**.** *The **complex projective space** $\mathbb{C}P^n$ is defined as follows: for all $\lambda \neq 0$,*

$$\begin{aligned}\mathbb{C}P^n &= \mathbb{C}^{n+1} \setminus \{0\}/\textit{dilations} \\ &= \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}/\big((z_0, \cdots, z_n) \sim (\lambda z_0, \cdots, \lambda z_n)\big) \\ &= S^{2n+1}/U(1)\end{aligned}$$

By definition, it is compact.

Given homogeneous polynomials $F_i(x_0, \cdots, x_n)$, we obtain a "projective variety"

$$X = X(F_1, \cdots, F_k) = \{x \in \mathbb{R}^k : F_i(x) = 0,\ \forall i\}.$$

As before, if the polynomial have real coefficients, the case becomes

$$X(\mathbb{R}) = \{x \in \mathbb{RP}^n : F_i(x) = 0,\ \forall i\}.$$

We can ask about the relation between the topology and geometry of $X(\mathbb{R})$ and $X(\mathbb{C})$ and the algebraic properties of X. For example, say X is the zero set of a single homogeneous polynomial F of degree d, can we recover d from $X(\mathbb{C})$ and $X(\mathbb{R})$? This is only a sensible question for irreducible F.

It turns out that $X_F(\mathbb{C})$ determines a homology class $[X_F(\mathbb{C})] \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z})$ and this group is cyclic with generator $[H]$ induced by $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ and $[X_F(\mathbb{C})] = d \cdot [H]$.

We can recover d from the intrinsic geometry at $X_F(\mathbb{C})$ using its "Chern class" in $H^*(X_F(\mathbb{C}))$. Over the real numbers, $H_{n-1}(\mathbb{R}P^b; \mathbb{Z}_2)$ is cyclic with generator $[H]$ and $[X_F(\mathbb{R})] = d \cdot [H]$, this gives us the recovery of d mod 2. It's possible to show that $X_F(\mathbb{R})$ does not provide an upper bound for d.

From a different point of view, the Nash embedding theorem shows that any smooth, closed manifold over $\mathbb{R}$ is diffeomorphic to $X(\mathbb{R})$ for some real, smooth, projective variety. For complex manifolds, the analogue statement is false.

To be diffeomorphic to a complex projective variety, a manifold must be complex, Kähler, Hodge and then it will have an embedding into $\mathbb{C}P^N$ for some integer N and Chow's theorem guarantees that it's algebraic. In this course, we'll show that compact submanifolds of $\mathbb{C}P^n$ satisfy these properties is a complex projective variety.

2 Complex Variables and holomorphic functions

2.1 Basic Settings

First recall some concepts and theorems for one complex variable. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function.

Definition 2.1 (holomorphic function)**.** *f is called **holomorphic** if f satisfies the Cauchy–Riemann equation. That is, write $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$, where $x, y \in \mathbb{R}$ and u, v are $\mathbb{R}$-valued C^1-functions, and they satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Or, $df(z)$ is $\mathbb{C}$-linear.

Definition 2.2 (analytic function)**.** *f is called **analytic** if for any $z_0 \in U$, there exists an $\epsilon > 0$ such that for every $z \in B_\epsilon(z_0)$, the ball with radius ϵ centered at z_0,*

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n.$$

Proposition 2.1. *f is analytic if and only if f is holomorphic. This is also equivalent to $f \in C^1$ satisfying the Cauchy integral formula: for every $z_0 \in U$, there exists a small $\epsilon > 0$ such that*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0} \, dz.$$

Let's introduce the differential operators

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

We justify the notations by

$$\partial_z z = \partial_{\bar{z}} \bar{z} = 1, \qquad \partial_z \bar{z} = \partial_{\bar{z}} z = 0.$$

In terms of the new notations, Cauchy–Riemann equation can be written by $\partial_{\bar{z}} f = 0$. *Geometrically,* $f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$, *then*

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}$$

induces $D_{z_0} f : T_{z_0}\mathbb{R}^2 \to T_{f(z_0)}\mathbb{R}^2$ *with respect to the standard bases. This is the* ***(real) Jacobian*** *of* f,

$$J_{\mathbb{R}}(f) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}.$$

Example 2.1.

$$(D_{(x_0,y_0)} f)(\partial_x) = \partial_t|_{t=0} f(x_0 + t, y_0) = \partial_t|_{t=0} \begin{bmatrix} u(x_0 + t, y) \\ v(x_0 + t, y) \end{bmatrix} = \begin{bmatrix} \partial_x u \\ \partial_x v \end{bmatrix}$$

After we complexify $D^C_{z_0} f : T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$, *we can write this matrix in the basis* $\partial_z, \partial_{\bar{z}}$ *for both domain and codomain:*

$$\begin{bmatrix} \partial_z(u + iv) & \partial_{\bar{z}}(u + iv) \\ \partial_z(u - iv) & \partial_{\bar{z}}(u - iv) \end{bmatrix} = \begin{bmatrix} \partial_z f & \partial_{\bar{z}} f \\ \partial_z \bar{f} & \partial_{\bar{z}} \bar{f} \end{bmatrix}.$$

Here we have $\partial_{\bar{z}} \overline{f} = \overline{\partial_z f}$ *and* $\overline{\partial_z \overline{f}} = \partial_{\bar{z}} f$. *The function* f *is holomorphic if and only if this matrix is diagonal. The* ***(complex) Jacobian*** *for* f *is*

$$J_{\mathbb{C}}(f) = \begin{bmatrix} \partial_z f \end{bmatrix}.$$

Holomorphic functions of one variable satisfy the following important theorems:

Theorem 2.1 (Maximum Principle)**.** *Suppose we have an open and connected set* $U \subset \mathbb{C}$ *and a holomorphic function* $f : U \to \mathbb{C}$ *that is non-constant. Then* $|f|$ *has no local maximum in* U. *If* U *is bounded and* f *can extend to a continuous function* $\tilde{f} : \overline{U} \to \mathbb{C}$, *then* $\max |f|$ *occurs on* $\partial \overline{U}$.

Theorem 2.2 (Identity Theorem)**.** *Suppose we have two holomorphic functions* $f, g : U \to \mathbb{C}$, *and* $U \subset \mathbb{C}$ *is connected. If* $\{z \in U : f(z) = g(z)\}$ *contains an open set, then it is all of* U.

Theorem 2.3 (Extension Theorem)**.** *Suppose we have a bounded holomorphic function* $f : B_\epsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ *defined on some ball of radius* $\epsilon > 0$ *centered at* z_0, *then it extends* *to a holomorphic function $\tilde{f} : B_\epsilon(z_0) \to \mathbb{C}$.*

Theorem 2.4 (Riemann Mapping Theorem)**.** *If $U \subset \mathbb{C}$ is a simply connected proper open set, then U is biholomorphic to the unit ball $B_1(0)$.*

Theorem 2.5 (Residue Theorem)**.** *If $f : B_\epsilon(0) \setminus \{0\} \to \mathbb{C}$ is holomorphic and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is its Laurent series, then*

$$a_{-1} = \frac{1}{2\pi i} \int_{\partial B_{\epsilon/2}(0)} f(z)\, dz.$$

We now begin the several complex variables part.

Definition 2.3 (holomorphic function with multiple complex variables)**.** *Let $U \subset \mathbb{C}^n$ be an open set, $f : U \to \mathbb{C}$ is continuous differentiable. Then f is* ***holomorphic*** *at $a = (a_1, \cdots, a_n) \in U$ if for all $j \in \{1, \cdots, n\}$, the function of one variable*

$$z_j \mapsto f(a_1, \cdots, a_{j-1}, z_j, a_{j+1}, \cdots, a_n)$$

is holomorphic at a_j, i.e. $\partial_{\overline{z_j}} f = 0$. If we write

$$(df)_{\mathbb{C}} = \underbrace{\sum \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j}}_{\overline{\partial} f},$$

then f is holomorphic if and only if $\overline{\partial} f = 0$.

Definition 2.4 (polydisc)**.** *For $a \in \mathbb{C}^n, R \in (\mathbb{R}^+)^n$, the* ***polydisc*** *around a with multi-radius R is the set*

$$D(a, R) = \{z \in \mathbb{C}^n : |z_j - a_j| < R_j,\ \forall j \in \{1, \cdots, n\}\}.$$

If $R = (1, \cdots, 1)$ and $a = 0$, we abbreviate $D(0, 1)$ by $\mathbb{D}^n$ and refer to it as the unit disc in $\mathbb{C}^n$.

Repeatedly applying the Cauchy formula in one variable, we obtain

Theorem 2.6. *Let $f : D(\omega, \epsilon) \to \mathbb{C}$ be a holomorphic function and $z \in D(\omega, \epsilon)$, then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D(\omega,\epsilon)} \frac{f(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)}\, d\xi_1 \cdots d\xi_n.$$

Using this theorem, we can show that for any $\omega \in U$, there exists $D(\omega, \epsilon) \subset U$ such that for all $z \in D(\omega, \epsilon)$,

$$f(z) = \sum_{|\alpha|=0}^{\infty} \frac{\partial_z^\alpha f}{\alpha!} (z - \omega)^\alpha.$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ *is a multi-index. To be explicit,*

$$(z-\omega)^\alpha = \prod_{k=1}^{n} (z_k - \omega_k),$$

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \qquad \partial_z^\alpha f = \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2} \cdots \partial_{z_n}^{\alpha_n} f.$$

From the list above, the Maximum Principle, the Identity Theorem, and the Liouville's Theorem generalize easily. Riemann Extension Theorem holds but is harder to prove. However, Riemann Mapping Theorem fails in several variables. There are also phenomena that do NOT have analogues in one complex variable. One example we shall see later is the Hartogs Extension Theorem.

2.2 Equation $\overline{\partial} u = f$

In this section, we will continue to list the counterexamples that do NOT have analogues in one complex variable. These are examples of the Hartogs phenomenon.

Example 2.2. *Consider*

$$H = \{(z, \omega) \in \mathbb{C}^2 : |z| < 1, \tfrac{1}{2} < |\omega| < 1\} \cup \{(z, \omega) \in \mathbb{C}^2 : |z| < \tfrac{1}{2}, |\omega| < 1\}.$$

Let f *be holomorphic on* H*. Claim: there exists a holomorphic function* F *defined on* $\mathbb{D} = \{(z, \omega) \in \mathbb{C}^2 : |z| < 1, |\omega| < 1\}$ *such that* $F|_H = f$*.*

In fact, we have for $r \in (\frac{1}{2}, 1)$

$$F(z, \omega) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(z, \xi)}{\xi - \omega} \, d\xi.$$

so F *is holomorphic. Indeed,*

$$\partial_{\overline{z}} \left(\frac{f(z, \xi)}{\xi - \omega} \right) = \partial_{\overline{\omega}} \left(\frac{f(z, \xi)}{\xi - \omega} \right) = 0.$$

For any fixed z *with* $|z| < \frac{1}{2}$, $\omega \mapsto f(z, \omega)$ *is holomorphic on all of* $\{(z, \omega) \in \mathbb{C}^2 : |\omega| < 1\}$*. So* $F(z, \omega) = f(z, \omega)$ *for any* $|z| < \frac{1}{2}, |\omega| < r$ *by the Cauchy integral formula, which implies* $F = f$ *on* H*.*

Lemma 2.1. *Let* $f \in C^1(\overline{\Omega})$*, then*

$$\int_{\partial\Omega} f dz = \int_\Omega \frac{\partial f}{\partial \overline{z}} \, d\overline{z} dz = 2i \int_\Omega \partial_{\overline{z}} f \, dxdy.$$

Proposition 2.2. *Let* $u \in C^1(\overline{\Omega})$*, then*

$$u(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\zeta)}{z - \zeta} dz + \frac{1}{\pi} \int_\Omega \frac{\partial u / \partial \overline{z}(z)}{z - \zeta} \, dxdy.$$

Proof. Fix ζ, let $\epsilon < d(\zeta, \partial\overline{\Omega})$ and let $\Omega_\epsilon = \{z \in \Omega : |z-\zeta| > \epsilon\}$. Apply Lemma 2.1 to $f(z) = \dfrac{u(z)}{z-\zeta}$, we obtain

$$\int_{\partial\Omega_\epsilon} \frac{u(z)}{z-\zeta}\, dz = 2i \int_{\Omega_\epsilon} \frac{\partial_{\overline{z}} u}{z-\zeta}\, dxdy.$$

As $\epsilon \to 0$, LHS converges to

$$-\int_{\partial\Omega} \frac{u(\zeta)}{z-\zeta}\, dz + 2\pi i\, u(\zeta),$$

which concludes our result. □

Theorem 2.7. *Let $\phi \in C_c^\infty(\mathbb{C})$ and*

$$u(\zeta) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(z)}{z-\zeta}\, dxdy,$$

then u is an analytic function outside of $\operatorname{supp}\phi$ and u is smooth on $\mathbb{C}$. Moreover, $\partial_{\overline{z}} u = \phi$.

Proof. Interchanging derivatives and the integral we see that $u \in C^\infty(\mathbb{C})$. By a change of variables, we have

$$u(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta - z)}{z}\, dxdy.$$

So

$$\partial_{\overline{\zeta}} u(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_{\overline{\zeta}} \phi(\zeta - z)}{z}\, dxdy = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_{\overline{z}} \phi(z)}{z-\zeta}\, dxdy.$$

Applying Proposition 2.2 to any disc containing $\operatorname{supp}\phi$, we get $\partial_{\overline{\zeta}} u = \phi$. □

Remark 2.1. *Even though ϕ has compact support, there is no solution of $\partial_{\overline{z}} u = \phi$ that can have compact support if $\int_{\mathbb{C}} \phi \neq 0$. Indeed, if $u(\zeta) = 0$ for any $|\zeta| > R$, then*

$$0 = \int_{|z|=R} u(z) dz = \int_{|z|<R} \partial_{\overline{z}} u\, dxdy = 2i \int_{|z|<R} \phi\, dxdy.$$

Theorem 2.8. *Suppose $f_j \in C_c^\infty(\mathbb{C}^n)$, $j \in \{1, \cdots, n\}, n > 1$, satisfy $\partial_{\overline{z_j}} f_k = \partial_{\overline{z_k}} f_j$ for every $j, k \in \{1, \cdots, n\}$. Then there is a $u \in C_c^\infty(\mathbb{C}^n)$ such that $\partial_{\overline{z_j}} u = f_j$ for every $j \in \{1, \cdots, n\}$.*

Proof. Define

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \cdots, z_n)}{\zeta - z_1}\, d\zeta d\overline{\zeta} = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(z_1 - \zeta, z_2, \cdots, z_n)}{\zeta}\, d\zeta d\overline{\zeta}.$$

We note that $u \in C^\infty(\mathbb{C})$. Since f_1 has compact support, u vanishes if $|z_2| + \cdots + |z_n| \gg$

0. By Theorem 2.7, $\partial_{\overline{z_1}} u = f_1$. Also differentiating, by Proposition 2.2,

$$\partial_{\overline{z_j}} u = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial_{\overline{z_1}} f_j(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} \, d\zeta d\overline{\zeta} = f_j.$$

Hence u solves the system of equations. Let $K = \bigcup_{j=1}^n \operatorname{supp} f_j$, then u is holomorphic on $\mathbb{C}^n \setminus K$. We know that u is zero if $|z_2| + \cdots + |z_n| \gg 0$, so by the Identity Theorem u must vanish on the unbounded component of $\mathbb{C}^n \setminus K$, which implies $u \in C_c^\infty(\mathbb{C}^\infty)$. □

2.3 Hartogs Extension Theorem

We will introduce the famous Hartogs Extension Theorem.

Theorem 2.9 (Hartogs Extension Theorem). *Let U be a domain in $\mathbb{C}^n$, $n > 1$; that is, U is a non-empty connected open set. Let K be a compact subset of U such that $U \setminus K$ is connected. Then every holomorphic function on $U \setminus K$ extends uniquely to a holomorphic function on U.*

Proof. Given an analytic function f defined on $U \setminus K$. Choose $\theta \in C_c^\infty(U)$ such that $\theta|_K = 1$. Define $f_0 \in C^\infty(U)$ by setting

$$f_0(z) = \begin{cases} 0, & z \in K, \\ (1-\theta) f, & z \in U \setminus K. \end{cases}$$

We shall construct $v \in C^\infty(\mathbb{C}^n)$ such that $f_0 + v$ is the required holomorphic extension of f. In order for $f_0 + v$ to be holomorphic we need

$$\partial_{\overline{z_j}}(f_0 + v) = \partial_{\overline{z_j}}(1-\theta) f + \partial_{\overline{z_j}} v = -(\partial_{\overline{z_j}} \theta) f + \partial_{\overline{z_j}} v,$$

that is, we need $\partial_{\overline{z_j}} v = (\partial_{\overline{z_j}} f)$ for every $j \in \{1, \cdots, n\}$. By Theorem 2.8, we can find $v \in C_c^\infty(\mathbb{C}^n)$ solving this system of equations. Since v has compact support and is holomorphic outside the support of θ, it must vanish on the unbounded component of $\mathbb{C}^n \setminus \operatorname{supp} \theta$. Since $\operatorname{supp} \theta \subset U$, there is an open set in $U \setminus K$ where $v \equiv 0$ and so $f_0 + v = f_0 = f$. But $U \setminus K$ is connected and $f, f_0 + v$ are holomorphic, so they must coincide on all of $U \setminus K$. Thus $f_0 + v$ is the desired holomorphic extension of f. □

Corollary 2.1. *Let $U \subset \mathbb{C}^n$ be a domain, $n > 1$, and f is holomorphic on U. The zero set $f^{-1}(0)$ of f is never a compact subset of U.*

Proof. Assume $K = f^{-1}(0)$ is compact and let $g : U \setminus K \subset \mathbb{C}^n \to \mathbb{C}^n$ be $g(z) = \dfrac{1}{f(z)}$. Then g is holomorphic on $U \setminus K$. Proceeding as in the proof of the Hartogs Extension Theorem, we pick $\theta \in C_c^\infty(U)$ with $\theta|_K = 1$. Define $g_0 \in C^\infty(U)$ to be 0 on K and $(1-\theta) g$ otherwise. We can find $v \in C_c^\infty(\mathbb{C}^n)$ such that v is holomorphic on

$\mathbb{C}^n \setminus \operatorname{supp}\theta$ and g_0+v is holomorphic on U. So v vanishes on the unbounded component of $U \setminus \operatorname{supp}\theta$. Thus there is an open set in $U \setminus K$ on which $g = g_0 = g_0 + v$ since g and $g_0 + v$ are holomorphic on $U \setminus K$, they coincide on the corresponding connected component of $U \setminus K$, say W. Finally, pick $(w_k) \subset W$, $w_k \to w_\infty \in K$. We have $|g(w_k)| = \dfrac{1}{|f(w_k)|} \to \infty$, but from $g(w_k) = (g_0+v)(w_k) \to (g_0+v)(w_\infty)$ we conclude $|g(w_k)|$ is bounded. Contradiction! $\square$

Heuristically, we might expect that the zero set of a nontrivial holomorphic function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ will have complex codimension 1. For example, if f is a polynomial of degree 1, then its zero set is an affine subspace of complex dimension $n-1$. If $D_p f : \mathbb{C}^n \to \mathbb{C}$ is nonzero at each $p \in f^{-1}(0)$, then f is "well-approximated" by its linear approximation and $f^{-1}(0)$ should be locally modeled by open subsets of $\mathbb{C}^{n-1}$. Indeed, the complex version of the implicit function theorem holds and shows that $f^{-1}(0)$ is a smooth submanifold of complex dimension $n-1$.

Things are more complicated if the derivative vanishes. In one complex variable, if nonzero function f satisfies $f(0) = 0$, then "0 is a root of a finite order" (say p) means that there is a holomorphic function g such that $g(0) \neq 0$ and $f(z) = z^p g(z)$. In several complex variables, after a change of coordinates, we can write the nontrivial function f to be $F(z_n) = f(0, z_n)$, where $0 \in \mathbb{C}^{n-1}$. So it has a zero of finite order, sat p, such that $F(z_n) = z_n^p g_0(z_n)$.

Using the continuity of f, we can apply Roche's Theorem to conclude that there is a polydisc $D(0,\epsilon) \subset \mathbb{C}^{n-1}$ such that for all $z' \in D(0,\epsilon)$, the function $z \mapsto f(z', z)$ has exactly p zeros in $D(0, \epsilon_n) \subset \mathbb{C}$. In particular, we see again that the zeros of a holomorphic function of several variables are not isolated.

2.4 Weierstrass Preparation Theorem

In this section, we will discussion what a holomorphic function looks like near a zero. In the one variable case, $f(z) = z^p g(z)$, where p is a positive integer and g is a holomorphic function with $g(0) \neq 0$ or $f \equiv 0$. Suppose we are given a holomorphic function $f(z_1, \cdots, z_{n-1}, \omega)$ near $0 \in \mathbb{C}^n$, and $f(0, \cdots, 0) = 0$, and ω-axis is not in $f^{-1}(0)$. Write $f_z(\omega) = f(z, \omega)$, where $z \in \mathbb{C}^{n-1}$, then $f_0(\omega)$ is not identically zero. We know that $f_0(\omega) = \omega^p g(\omega)$, where $g(0) \neq 0$, p is a positive integer. There exists a $r > 0$ such that $|f_0(\omega)| > \delta > 0$ whenever $|\omega| = r$. So there exists an $\epsilon > 0$ such that $|z| < \epsilon$, $|\omega| = r$, then $|f_z(\omega)| \geq \dfrac{\delta}{2} > 0$.

Writing $f_z(\omega) = \tilde{f}_z(\omega) \prod_{j=1}^p (\omega - a_j(z))$, we see that

$$\sum_j a_j(z)^q = \frac{1}{2\pi i} \int_{|\omega|=r} \omega^q \frac{f_z'(\omega)}{f_z(\omega)}\, d\omega.$$

This shows that the LHS is a holomorphic function of z for any q. Hence the elementary symmetric functions of the $a_j(z)$ are holomorphic functions of z. Denote them by $\sigma_j(z)$. Then $g_z(\omega) = \omega^p - \sigma_1(z)\omega^{p-1} + \cdots + (-1)^N \sigma_N(z) = \prod(\omega - a_j(z))$ is also holomorphic function of z. So, $g(z,\omega) = g_z(\omega)$ is holomorphic on $\{(z,\omega) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| < \epsilon,\ |\omega| < r\}$. It also has the same zeros as $f(z,\omega)$ on this set.

Define

$$h(z,\omega) = \frac{f(z,\omega)}{g(z,\omega)}.$$

One can check it is well-defined and holomorphic off the zero set. Fix z, $h_z(\omega)$ has only removable singularities so it extends to a holomorphic on $D(0,\epsilon) \times D(0,r)$. The extended one is holomorphic in ω for each z and holomorphic off the zero set. Writing

$$h(z,\omega) = \frac{1}{2\pi i} \int_{|u|=r} \frac{h(z,u)}{u-\omega}\, du,$$

we see that h is holomorphic in z.

Definition 2.5 (Weierstrass polynomial)**.** *A* ***Weierstrass polynomial*** *in ω is a polynomial of the form*

$$\omega^p + \alpha_1(z)\omega^{p-1} + \cdots + \alpha_p(z),$$

where $\alpha_j(z)$ is holomorphic for each j with $\alpha_j(0) = 0$.

Theorem 2.10 (Weierstrass Preparation)**.** *Let f be a holomorphic function near the origin in $\mathbb{C}^n$ and $f(0) = 0$. f is also assumed not to be identically zero on the ω-axis. Then there is a neighborhood of* 0 *in which f can be written uniquely as $f = g \cdot h$, where g is a Weierstrass polynomial of degree p in ω and $h(0) \neq 0$.*

Theorem 2.11 (Riemann Extension)**.** *Suppose $f(z,\omega)$ is holomorphic in a ball $\mathbb{B} \subset \mathbb{C}^n$, and $g(z,\omega)$ is holomorphic on $\mathbb{B} \setminus f^{-1}(0)$ and bounded. Then g extends to a holomorphic function on $\mathbb{B}$.*

Proof. WLOG, assume that ω-axis is not contained in $f^{-1}(0)$. As before, there are r, ϵ such that $|f(z,\omega)| > \delta > 0$ whenever $|z| < \epsilon$, $|\,\omega\,| = r$. The one variable version of Riemann extension then applies to each g_z and the extension $\tilde{g}_z$ satisfies

$$\tilde{g}_z(\omega) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{g_z(\xi)}{\xi - \omega}\, d\xi.$$

Hence $\tilde{g}(z,\omega) = g_z(\omega)$ is holomorphic in (z,ω) for all $|z| < \epsilon$, $|\,\omega\,| < r$. ∎

Now, we will discuss the failure of Riemann mapping theorem in several complex variables.

Example 2.3. *Consider $H = \{z \in \mathbb{C}^n :\ \Im z_1 > 0\}$ and $\mathbb{B}^n = \{z \in \mathbb{C}^n :\ |z| < 1\}$. If $\psi : H \to \mathbb{B}^n$ is holomorphic, then for each z_1 with $\Im z_1 > 0$, the function $(z_2, \cdots, z_n) \mapsto \psi(z_1, z_2, \cdots, z_n)$ is holomorphic and bounded on $\mathbb{C}^{n-1}$. Hence it is constant.*

Theorem 2.12 (Poincaré theorem). *For $n > 1$, the unit polydisc $\mathbb{D}^n$ and the unit ball $\mathbb{B}^n$ are not biholomorphic.*

Proof. Assume $\psi : \mathbb{D}^n \to \mathbb{B}^n$ is a biholomorphic with $\psi(0) = 0$ and let $\Phi : \mathbb{B}^n \to \mathbb{D}^n$ be the inverse. We claim that we must have $D_0\psi(\mathbb{D}^n) \subset \mathbb{B}^n$ and $D_0\Phi(\mathbb{B}^n) \subset \mathbb{D}^n$. Given the claim we have $D_0\psi(\mathbb{D}^n) = \mathbb{B}^n$ and $D_0\psi(\partial\mathbb{D}^n) = \partial\mathbb{B}^n$. This is impossible since $D_0\psi$ is linear and $\partial\mathbb{D}^n$ contains linear pieces of positive dimension, which $\partial\mathbb{B}^n$ does not.

Now to prove the claim.

1. $D_0\psi(\mathbb{D}^n) \subset \mathbb{B}^n$:

   Write $\psi = (\psi_1, \cdots, \psi_n)$, where $r \in \mathbb{D}^n$ and $u = (u_1, \cdots, u_n) \in \mathbb{B}^n$. Applying Schwarz's lemma to the function

   $$t \mapsto \sum_{j=1}^{n} u_j \psi_j(tv),$$

   we see that $|\langle \bar{u}, (D_0\psi)(v)\rangle| \leq 1$. As this holds for all $u \in \mathbb{B}^n$, we must have $|D_0\psi(v)| \leq 1$.

2. $D_0\Phi(\mathbb{B}^n) \subset \mathbb{D}^n$:

   Write $\Phi = (\Phi_1, \cdots, \Phi_n)$ and $u = (u_1, \cdots, u_n) \in \mathbb{B}^n$. Applying Schwarz's lemma to the function

   $$t \mapsto \Phi_j(tu_1, \cdots, tu_n),$$

   we see that $|\sum u_k \, \partial_{z_k} \Phi_j(0)| \leq 1$ for $1 \leq j \leq n$. Hence $D_0\Phi(u) \in \mathbb{D}^n$.

$\square$