This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 Group representations, Maschke's theorem, characters

1.1 Finite group representations

For a $\mathbb{K}$-vector space V recall that $GL(V)$ is the group of invertible linear maps $V \to V$.

Definition 1.1 (group representation)**.** Let G be a group. A group representation of G is a pair (V, ρ) where V is a vector space and $\rho : G \to GL(V)$ is a group homomorphism. In earlier lectures, we saw that representations of G are the same thing as representations of the group algebra $\mathbb{K}[G] = \mathbb{K}\text{-span}\{a_g : g \in G\}$.

From now on, we will think of elements of $\mathbb{K}[G]$ as formal (finite) linear combinations of group elements, writing $\sum_{g\in G} c_g g$ instead of $\sum_{g\in G} c_g a_g$, where $c_g \in \mathbb{K}$ and a_g is the formal symbol indexed by $g \in G$.

We are interested in representations of **finite** groups G. In this case $\mathbb{K}[G]$ has finite dimension.

Our first important question to answer is: when is $\mathbb{K}[G]$ semisimple?

From this point on, assume that the group G is finite. Write $|G|$ for its number of elements.

Theorem 1.2 (Maschke's theorem)**.** Assume that $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then $\mathbb{K}[G]$ is semisimple.

Proof. Let (V, ρ) be a finite-dimensional G-representation, and hence also a $\mathbb{K}[G]$-representation.

It suffices to check that V is a direct sum of irreducible subrepresentations. This clearly holds if (V, ρ) is irreducible so assume this is not the case. Then V must have an irreducible subrepresentation W by one of our homework exercises. By induction on dimension, it is enough to show that V has another nonzero subrepresentation U such that $V = W \oplus U$.

We can find a subspace $\tilde{U}$, not necessarily a subrepresentation, with $V = W \oplus \tilde{U}$ as vector spaces.

Just choose a basis $w_1, w_2, \dots, w_m$ of W, extend this to a basis $w_1, \dots, w_m, u_1, \dots, u_n$ for V, and set

$$\tilde{U} = \mathbb{K}\text{-span}\{u_1, \dots, u_n\}.$$

Here is the key idea to the proof.

Let $\pi : V \to W$ be linear map with $\pi(w_i) = w_i$ for all i and $\pi(u_j) = 0$ for all j. Then define

$$\sigma = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1})$$

Finally consider $U = \mathrm{kernel}(\sigma)$. We claim that:

1. U is a subrepresentation.
2. $V = W \oplus U$.

Property (1) holds because for any $h \in G$ we have

$$\sigma\rho(h) = \frac{1}{|G|} \sum_{g\in G} \rho(g) \circ \pi \circ \rho(g^{-1}h) = \frac{1}{|G|} \sum_{x\in G} \rho(hx) \circ \pi \circ \rho(x^{-1}) = \rho(h)\sigma,$$

making the substitution $x = h^{-1}g$ in the second equality.

Thus $\sigma(u) = 0$ if and only if $\sigma\rho(h)(u) = \rho(h)\sigma(u) = 0$ for any $h \in G$ and $u \in U$, as $\rho(h)$ is invertible. For property (2), note that $\rho(W) \subseteq W$ and $\pi(w) = w$ for all $w \in W$, so $\sigma(w) = w$ for all $w \in W$. Since $\sigma(V) \subseteq W$, it follows that $\sigma^2 = \sigma$. Thus any $v \in V$ can be written as

$$v = \sigma(v) + (v - \sigma(v))$$

where $\sigma(v) \in W$ and $(v - \sigma(v)) \in U$, and we have $W \cap U = 0$ since if $x \in W \cap U$ then $x = \sigma(x) = 0$. Thus $V = W \oplus U$ as needed. $\square$

Corollary 1.3. Assume $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. Then there are finitely many isomorphism classes of irreducible G-representations $\{(V_i, \rho_i)\}_{i \in I}$, all of which have finite dimension, and we have

$$|G| = \sum_{i \in I} (\dim V_i)^2 \quad \text{and} \quad \mathbb{K}[G] \cong \bigoplus_{i \in I} \mathrm{End}(V_i).$$

The representation theory of finite-dimensional semisimple algebras is trivial in the sense that everything is just a direct sum of matrix algebras. What makes the representation theory of finite groups more interesting is the distinguished basis of $\mathbb{K}[G]$ provided by G itself. Going from this basis to the natural bases of $\mathbb{K}[G]$ viewed as a sum of matrix algebras is nontrivial.

It turns out that the converse to Maschke's theorem is also true.

Theorem 1.4 (Converse to Maschke's theorem)**.** If $\mathbb{K}[G]$ is semisimple then $\mathrm{char}(\mathbb{K})$ does not divide $|G|$.

Proof. Assume $\mathbb{K}[G]$ is semisimple and consider the subspace

$$U \overset{\text{def}}{=} \mathbb{K}\text{-span}\left\{\sum_{g \in G} g\right\}.$$

This is a 1-dimensional subrepresentation of $\mathbb{K}[G]$.

By semisimplicity, there exists a complementary subrepresentation $V \subset \mathbb{K}[G]$ with $\mathbb{K}[G] = U \oplus V$. View $\mathbb{K}$ as a G-representation with $g \cdot c = c$ for all $g \in G$ and $c \in \mathbb{K}$.

Then define $\phi : \mathbb{K}[G] \to \mathbb{K}$ to be linear map that sends $V \to 0$ and $\sum_{g \in G} g \mapsto 1_{\mathbb{K}}$. Because U and V are subrepresentations, the map ϕ is a morphism of $\mathbb{K}[G]$-representations. Thus $\phi(g) = \phi(g \cdot 1_G) = g \cdot \phi(1_G) = \phi(1_G) \in \mathbb{K}$ for all $g \in G$.

But this means that

$$1_{\mathbb{K}} = \phi(\sum_{g \in G} g) = \sum_{g \in G} \phi(g) = \sum_{g \in G} \phi(1_G) = |G|\phi(1_G).$$

Thus $|G|$ is invertible (and nonzero) in $\mathbb{K}$, so $\mathrm{char}(\mathbb{K})$ must not divide $|G|$. $\square$

1.2 Characters of group representations

Continue to let G be a finite group.

If (V, ρ) is a G-representation with $\dim V < \infty$ then its *character* is the map $\chi_{(V,\rho)} : G \to \mathbb{K}$ with

$$\chi_{(V,\rho)}(g) = \mathrm{tr}(\rho(g)).$$

Since traces are invariant under change of basis, it follows that:

Fact 1.5. If $(V, \rho) \cong (V', \rho')$ as G-representations then $\chi_{(V,\rho)} = \chi_{(V',\rho')}$.

The **conjugacy classes** of G are the sets $K_g \overset{\text{def}}{=} \{xgx^{-1} : x \in G\}$ for $g \in G$.

A **class function** of G is a map $G \to \mathbb{K}$ that is constant on all elements in each conjugacy class.

Equivalently, $f : G \to \mathbb{K}$ is a class function if and only if $f(xgx^{-1}) = f(g)$ for all $x, g \in G$.

Fact 1.6. The character of any finite-dimensional G-representation is a class function.

We say that the character $\chi_{(V,\rho)}$ is *irreducible* if (V, ρ) is an irreducible representation.

We mention some special properties of irreducible characters that hold when $\mathbb{K}[G]$ is semisimple.

Proposition 1.7. If $\mathrm{char}(\mathbb{K})$ does not divide $|G|$ then the irreducible characters of G are a basis for the vector space of class functions of G.

Proof. In this case $\mathbb{K}[G]$ is semisimple so the irreducible characters are a basis for $(\mathbb{K}[G]/[\mathbb{K}[G], \mathbb{K}[G]])^*$.

By definition, this dual space can be identified with the vector space of linear maps $f : \mathbb{K}[G] \to \mathbb{K}$ that satisfy $f(XY) = f(YX)$ for all $X, Y \in \mathbb{K}[G]$. Check that this is the same as the set of linear maps $f : G \to \mathbb{K}$ with $f(gh) = f(hg)$ for all $g, h \in G$, or equivalently with $f(xgx^{-1}) = f(g)$ for all $x, g \in G$. Thus, we can identify $(\mathbb{K}[G]/[\mathbb{K}[G], \mathbb{K}[G]])^*$ with the vector space of class functions of G. $\square$

Corollary 1.8. If $|G|$ is not divisible by $\mathrm{char}(\mathbb{K})$ then the number of isomorphism classes of irreducible G-representations is the same as the number of distinct irreducible characters of G, which is also the number of distinct conjugacy classes of G.

Corollary 1.9. If $\mathrm{char}(\mathbb{K}) = 0$ then two finite-dimensional G-representations are isomorphic if and only if they have the same character.

A group G is **abelian** if $gh = hg$ for all $g, h \in G$. This holds if and only if the group algebra $\mathbb{K}[G]$ is commutative, so the following is true:

Fact 1.10. If G is abelian then all irreducible G-representations are 1-dimensional.

Suppose $f : V \to W$ is a linear map between vector spaces.

Recall that V^* is the vector space of linear maps $\lambda : V \to \mathbb{K}$.

Define $f^* : W^* \to V^*$ to be the linear map with the formula $f^*(\lambda) = \lambda \circ f$.

If $f \in GL(V)$ then $f^* \in GL(V^*)$ since $(f \circ g)^* = g^* \circ f^*$.

Now suppose (V, ρ_V) is a G-representation. Define $\rho_{V^*} : G \to GL(V^*)$ by the formula

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

Fact 1.11. If (V, ρ_V) is a representation then so is (V^*, ρ_{V^*}).

From this point on, we assume $\dim V < \infty$.

Fact 1.12. We have $\mathrm{tr}(f) = \mathrm{tr}(f^*)$ so $\chi_{(V^*,\rho_{V^*})}(g) = \chi_{(V,\rho)}(g^{-1})$ for all $g \in G$.

Since G is a finite group, any $g \in G$ has $g^{|G|} = 1_G$, and so any eigenvalue of $\rho_V(g)$ is a root of unity.

The character value $\chi_{(V,\rho_V)}(g)$ is the sum of the eigenvalues of $\rho_V(g)$, and is therefore a sum of roots of unity in $\mathbb{K}$. When $\mathbb{K} = \mathbb{C}$, the inverse of any root of unity is its complex conjugate.

As the eigenvalues of $\rho_V(g^{-1})$ are the inverses of the eigenvalues of $\rho_V(g)$, we deduce that:

Fact 1.13. If $\mathbb{K} = \mathbb{C}$ then $\overline{\chi_{(V,\rho_V)}(g)} = \chi_{(V,\rho_V)}(g^{-1}) = \chi_{(V^*,\rho_{V^*})}(g)$ for all $g \in G$. In this case $(V, \rho_V) \cong (V^*, \rho_{V^*})$ if and only if $\chi_{(V,\rho_V)}$ takes all **real** values.

Finally suppose (V, ρ_V) and (W, ρ_W) are G-representations. Then $(V \otimes W, \rho_{V\otimes W})$ is a G-representation when $\rho_{V\otimes W}(g)$ is linear map sending $v \otimes w \mapsto \rho_V(g)(v) \otimes \rho_W(g)(w)$ for $g \in G$, $v \in V$, and $w \in W$.

Fact 1.14. If $\dim V < \infty$ and $\dim W < \infty$ then $\chi_{(V\otimes W,\rho_{V\otimes W})} = \chi_{(V,\rho_V)}\chi_{(W,\rho_W)}$.

Remark 1.15. A G-representation is a left $\mathbb{K}[G]$-module. The algebra $\mathbb{K}[G]$ is often noncommutative.

Earlier, we emphasized that if A is a noncommutative algebra then the tensor product of two left A-modules is not a well-defined left A-module in general.

So how do we explain the existence of a tensor product for group representations?

Solution: the tensor product of two left A-modules does have the structure of a left $A \times A$-module. In particular, the tensor product of (V, ρ_V) and (W, ρ_W) is a representation of $\mathbb{K}[G] \otimes \mathbb{K}[G]$.

A special property of group algebras is that $\mathbb{K}[G] \otimes \mathbb{K}[G]$ has a subalgebra $\mathbb{K}\text{-span}\{g \otimes g : g \in G\} \cong \mathbb{K}[G]$. By identifying $\mathbb{K}[G]$ with this subalgebra, any $\mathbb{K}[G] \otimes \mathbb{K}[G]$-representation can be viewed as a $\mathbb{K}[G]$-representation, and this is how we define the G-representation $(V, \rho_V) \otimes (W, \rho_W)$.

2 Orthogonality relations, Unitary representations

2.1 More special properties of characters

For the rest of today, we assume that G is a finite group.

Suppose V and W are G-representations. Let $\mathrm{Hom}_{\mathbb{K}}(W, V)$ denote the set of linear maps $W \to V$.

The vector space $\mathrm{Hom}_{\mathbb{K}}(W, V)$ is a left $\mathbb{K}[G] \otimes \mathbb{K}[G]$-module for the action

$$(g \otimes h) \cdot \varphi : w \mapsto g\varphi(h^{-1}w) \quad \text{for } g, h \in G.$$

Indeed, notice that if $\phi : W \to V$ is linear and $w \in W$ then

$$((g_1g_2 \otimes h_1h_2) \cdot \varphi)(w) = g_1g_2\varphi(h_2^{-1}h_1^{-1}w) = g_1(g_2 \otimes h_2 \cdot \varphi)(h_1^{-1}w) = ((g_1 \otimes h_1)(g_2 \otimes h_2) \cdot \varphi)(w)$$

for any $g_1, g_2, h_1, h_2 \in G$. Now assume that V and W are finite-dimensional.

Proposition 2.1. It holds that $V \otimes W^* \cong \mathrm{Hom}_{\mathbb{K}}(W, V)$ as $\mathbb{K}[G] \otimes \mathbb{K}[G]$-modules.

Proof. Let $F : V \otimes_{\mathbb{K}} W^* \to \mathrm{Hom}_{\mathbb{K}}(W, V)$ be the linear map sending

$$v \otimes \varphi \mapsto (w \mapsto \varphi(w)v) \quad \text{for } v \in V \text{ and } \varphi \in W^*.$$

Notice that if $\{v_i\}$ is a basis for V, $\{w_j\}$ is basis for W, and $\{\delta_j\}$ is the dual basis for W^*, then F sends $v_i \otimes \delta_j$ to the linear map $W \to V$ whose matrix in the chosen bases has a one in position (i, j) and zeros elsewhere. Any linear map $W \to V$ is a linear combination of such images $F(v_i \otimes \delta_j)$, so F is surjective.

Because $\dim(V \otimes_{\mathbb{K}} W^*) = \dim V \dim W^* = \dim V \dim W = \dim(\mathrm{Hom}_{\mathbb{K}}(W, V))$, as V and W are finite-dimensional, the map F is an isomorphism of $\mathbb{K}$-vector spaces.

For any $g, h \in G$, $v \in V$, $w \in W$, and $\varphi \in W^*$, we have

$$((g \otimes h) \cdot F(v \otimes \varphi))(w) = g\varphi(h^{-1}w)v$$

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{\varphi} & V \end{array}.$$

and

$$F((g\otimes h)\cdot(v\otimes\varphi))(w) = F(gv\otimes\varphi\circ h^{-1})(w) = \varphi(h^{-1}w)(gv) = g\varphi(h^{-1}w)v.$$

Hence, F is an isomorphism of $\mathbb{K}[G]\otimes\mathbb{K}[G]$-modules. $\square$

By letting $g\in G$ act as $g\otimes g$, we can view $V\otimes_{\mathbb{K}[G]} W^* \cong \mathrm{Hom}_{\mathbb{K}}(W,V)$ as isomorphic G-modules.

Proposition 2.2. The set $(\mathrm{Hom}_{\mathbb{K}}(W,V))^G$ of elements in $\mathrm{Hom}_{\mathbb{K}}(W,V)$ fixed by all $g\in G$ is $\mathrm{Hom}_G(W,V)$.

Proof. Notice that if $\varphi\in\mathrm{Hom}_G(W,V)$, then for any $g\in G$, we have the following commutative diagram
Since the vertical map is invertible, we have $\varphi(w) = g(\varphi(g^{-1}w)) = (g\cdot\varphi)(w)$ for any $w\in W$. Thus, $\mathrm{Hom}_G(W,V)\subseteq(\mathrm{Hom}_{\mathbb{K}}(W,V))^G$.

Conversely, if $\varphi\in(\mathrm{Hom}_{\mathbb{K}}(W,V))^G$, then for any $g\in G$ and $w\in W$, we have

$$\varphi(gw) = (g\cdot\varphi)(gw) = g\varphi(g^{-1}gw) = g\varphi(w).$$

Thus, $\varphi\in\mathrm{Hom}_G(W,V)$ and $(\mathrm{Hom}_{\mathbb{K}}(W,V))^G\subseteq\mathrm{Hom}_G(W,V)$. $\square$

Combining the preceding results lets us deduce that:

Corollary 2.3. There is an isomorphism $(V\otimes_{\mathbb{K}[G]} W^*)^G\cong\mathrm{Hom}_G(W,V)$ as G-modules.

From now on, we assume $\mathbb{K}=\mathbb{C}$.

For any maps $f_1, f_2 : G\to\mathbb{C}$, we define a positive-definite Hermitian form

$$(f_1, f_2) := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Fact 2.4. When V is finite dimensional, we have $\overline{\chi_V^*(g)} = \chi_V(g^{-1})$ for any $g\in G$. If we further assume that $\mathbb{K}=\mathbb{C}$, then $\chi_{V^*}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g\in G$.

Theorem 2.5. The set $\mathrm{Irr}(G)$ is an orthonormal basis for the class functions on G with respect to $(\cdot,\cdot)$. In other words, we have $(\chi,\psi)=\delta_{\chi\psi}$ for any $\chi,\psi\in\mathrm{Irr}(G)$.

Proof. By Schur's Lemma, it suffices to prove that for any G-representations V and W, we have

$$(\chi_V,\chi_W) = \dim\mathrm{Hom}_G(W,V).$$

Let $\pi := \frac{1}{|G|}\sum_{g\in G} g\in\mathbb{K}[G]$. By Fact 2.4, we have

$$(\chi_V,\chi_W) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_{W^*}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes_{\mathbb{K}[G]}W^*}(g) = \chi_{V\otimes_{\mathbb{K}[G]}W^*}(\pi).$$

If X is any G-representation, then $X^G := \{x\in X : gx = x\}$ is a subrepresentation of G. Notice that $g\pi = \frac{1}{|G|}\sum_{h\in G} gh = \frac{1}{|G|}\sum_{gh\in G} gh = \pi$ for any $g\in G$.

Therefore, we have $\pi x \in X^G$ for any $x \in X$ and $\pi : X \twoheadrightarrow X^G$ is a projection map. Thus $\dim(X^G) = \chi_X(\pi)$. Restricting to the case when $X = V \otimes_{\mathbb{K}[G]} W^*$, we get

$$\chi_{V \otimes_{\mathbb{K}[G]} W^*}(\pi) = \dim(V \otimes_{\mathbb{K}[G]} W^*)^G = \dim(\mathrm{Hom}_G(W, V))$$

by Corollary 2.3. $\square$

For $g \in G$, let $Z_g := \{h \in G : hgh^{-1} = g\}$ be the centralizer of g.

Also let $K_g := \{hgh^{-1} : h \in G\}$ be the conjugacy class of g.

Fact 2.6. By the *Orbit-Stabilizer Theorem* it holds that $|K_g| = \frac{|G|}{|Z_g|}$.

Theorem 2.7. Let $g, h \in G$. Then

$$\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} = |Z_g| K_g = K_h,$$
$$0 K_g \neq K_h.$$

Proof. (proof sketch)

We want to describe this sum as the trace of a $\mathbb{C}$-endomorphism of $\mathbb{C}[G]$.

If we write V_ψ for an irreducible representation with character ψ, then we have

$$\begin{aligned}
\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)} &= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi}(g)\chi_{V_\psi^*}(h) \\
&= \sum_{\psi \in \mathrm{Irr}(G)} \chi_{V_\psi \otimes V_\psi^*}(g \otimes h) \\
&= \mathrm{tr}\left(\left(\bigoplus_{\psi \in \mathrm{Irr}(G)} \rho_{V_\psi \otimes V_\psi^*}\right)(g \otimes h)\right)
\end{aligned}$$

We have an isomorphism $\bigoplus_{\psi \in \mathrm{Irr}(G)} V_\psi \otimes V_\psi^* \cong \bigoplus_{\psi \in \mathrm{Irr}(G)} \mathrm{End}(V_\psi) \cong \mathbb{C}[G]$ of $\mathbb{C}[G] \otimes \mathbb{C}[G]$ representations.

Under this isomorphism, $g \otimes h$ acts on $\mathbb{C}[G]$ as the linear map sending $x \in G$ to gxh^{-1}.

Thus $\sum_{\psi \in \mathrm{Irr}(G)} \psi(g)\overline{\psi(h)}$ is the trace of $x \mapsto gxh^{-1}$, which is

$$|\{x \in G : x = gxh^{-1}\}| = |\{x \in G : g = xhx^{-1}\}| = \begin{cases} |Z_g| & \text{if } K_g = K_h \\ 0 & \text{if } K_g \neq K_h. \end{cases}$$

$\square$

2.2 Unitary representations

Definition 2.8 (unitary representation)**.** A finite-dimensional representation (V, ρ) of a group G (over $\mathbb{C}$) is **unitary** if there is a G-invariant positive definite Hermitian form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ with

$$(\rho(g)v, \rho(g)w) = (v, w) \text{ for any } v, w \in V \text{ and } g \in G.$$

Proposition 2.9. If G is a finite group, then any finite dimensional G-representation is unitary.

Proof. Pick any basis $\{v_i\}$ for V. We define a positive-definite Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with

$$\langle v_i, v_j \rangle = 1 \text{if } i = j \\ 0 \text{if } i \neq j.$$

Then the form $(v_i, v_j) := \sum_{g \in G} \langle gv_i, gv_j \rangle$ is positive-definite and Hermitian. □

Proposition 2.10. If (V, ρ) is a finite-dimensional unitary representation of a (not necessarily finite) group G, then (V, ρ) is semisimple.

Proof. Any irreducible representation is semisimple so assume V is reducible. Choose an irreducible subrepresentation of $U \subseteq V$. Write $(\cdot, \cdot)$ for the form that makes V unitary.

Let $U^\perp = \{v \in V : (v, u) = 0 \text{ for all } u \in U\}$. Then $V = U \oplus U^\perp$ and $U^\perp$ is a subrepresentation since the relevant form is G-invariant, so the result follows by induction on dimension. □

2.3 Matrix elements

Continue to assume that G is a finite group and V is a finite dimensional irreducible $\mathbb{C}[G]$-module.

Choose a G-invariant positive definite Hermitian form $(\cdot, \cdot)$ on V and let $\{v_i\}$ be an orthonormal basis on V with respect to $(\cdot, \cdot)$. Define $t_{ij}^V(g) := (gv_i, v_j)$ for any $g \in G$.

For each pair (i, j) with $1 \leq i, j \leq \dim V$, the map $t_{ij}^V : G \to \mathbb{C}$ is called a **matrix element**.

Proposition 2.11. The rescaled matrix elements $\frac{1}{\sqrt{\dim V}} t_{ij}^V : G \to \mathbb{C}$ (as V ranges over all isomorphism classes of finite dimensional irreducible G-representations and i, j range over the indices of an orthonormal basis of V) give an orthonormal basis of the space of *all* functions $G \to \mathbb{C}$.

We won't present the proof in class, but this can be found in the textbook.

Note that number of distinct matrix elements is $\sum_V (\dim V)^2 = |G|$.

2.3.1 Character tables

Suppose G is a finite group. Choose representatives $1 = g_1, g_2, \cdots, g_r$ for distinct conjugacy classes in G.

Suppose $\mathbf{1} = \chi_1, \chi_2, \cdots, \chi_r$ are the distinct elements in $\mathrm{Irr}(G)$.

Here $\mathbf{1}$ denotes the irreducible character $G \to \{1\}$.

Then everything you want to know about $\mathrm{Irr}(G)$ is encoded by the matrix

$\mathrm{Irr}(G)$	$1 = g_1$	g_2	$\cdots$	g_r
$\mathbf{1} = \chi_1$	1	1	$\cdots$	1
χ_2	$\chi_2(1)$	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_r	$\chi_r(1)$	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

called the *character table* of G.

Example 4.2. If $G = S_3$, then the character table of G is

$\mathrm{Irr}(S_3)$	1	$(1,2)$	$(1,2,3)$
$1 = \chi_{(3)}$	1	1	1
$\chi_{(2,1)}$	2	0	-1
$\chi_{(1,1,1)}$	1	-1	1

Using the character table orthogonality relations from today, you can compute the sizes of all conjugacy classes in G. Then you can decompose arbitrary products of characters into irreducibles.