This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 semisimple representations, density theorem

In this lecture, we begin a new chapter focusing on some general results about algebra representations.

1.1 Basic definitions and examples

From now on, we will assume that K is an algebraically closed field, and A is a K-algebra.

Definition 1.1 (semisimple representation)**.** representation of A is **semisimple** or **completely reducible** if it is isomorphic to a direct sum of irreducible representations.

As a general rule in mathematical terminology:

$$\text{“simple”} \equiv \text{“irreducible”} \qquad \text{and} \qquad \text{“semisimple”} \equiv \text{“(direct) sum of simple objects”}.$$

Notation. Suppose V is a left A-module. Often we will say that "V is a representation of A": this just means the representation (V, ρ) where $\rho : A \to \mathrm{End}(V)$ is defined by $\rho(a) : x \mapsto ax$ for $a \in A$ and $x \in V$.

Example 1.2 (Matrix algebras)**.** Let $A = \mathrm{Mat}_n(K)$ be the algebra of $n \times n$ matrices over K and let $V = K^n$ be the K-vector space of column vectors with n rows.

We can transform any vector in V by multiplying it on the left by a matrix in A, and this makes V into an A-representation: in other words, given $X \in A$ and $v \in V$ let Xv just mean matrix multiplication.

This representation is irreducible since if $v, w \in W$ and $v \neq 0$ then some $X \in A$ has $Xv = w$. So every nonzero vector is **cyclic** in the sense that it is not contained in any proper A-subrepresentation, proof by the **orbit** of the endomorphism algebra.

In this case we have $\mathrm{End}(V) = A$, which is also an A-representation, via the *regular representation* in which one matrix acts on another by ordinary matrix multiplication $X : Y \mapsto XY$.

The regular representation of A is semisimple as we have $A \cong V^{\oplus n}$ as A-representations.

An explicit isomorphism $A \xrightarrow{\sim} V^{\oplus n}$ is the map sending

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix} \mapsto \left(\begin{bmatrix} X_{11} \\ \vdots \\ X_{n1} \end{bmatrix}, \begin{bmatrix} X_{12} \\ \vdots \\ X_{n2} \end{bmatrix}, \cdots, \begin{bmatrix} X_{1n} \\ \vdots \\ X_{nn} \end{bmatrix} \right).$$

Notation. Here we define $V^{\oplus n}$ to be the set of n-tuples $(v_1, v_2, \ldots, v_n)$ where each $v_i \in V$ and where

$$(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \overset{\text{def}}{=} (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n),$$
$$c(v_1, v_2, \ldots, v_n) \overset{\text{def}}{=} (cv_1, cv_2, \ldots, cv_n),$$

for $v_i, w_i \in V$ and $c \in K$.

Example 1.3. More generally, suppose A is any algebra and V is an irreducible A-representation of finite dimension n. Then $\mathrm{End}(V) = \{\text{linear maps } L : V \to V\}$ is an A-representation for the action

$$a \cdot L : v \mapsto a \cdot L(v) \quad \text{for } a \in A \text{ and } v \in V.$$

This representation is semisimple with $\mathrm{End}(V) \cong V^{\oplus n}$ as A-representations. If V has basis $\{v_1, \cdots, v_n\}$ then an explicit isomorphism $\mathrm{End}(V) \xrightarrow{\sim} V^{\oplus n}$ is provided by the map $L \mapsto (L(v_1), \cdots, L(v_n))$

1.2 Subrepresentation of semisimple representations

Our main results today are derived from the following technical property. Among other consequences, it tells us that all subrepresentations of semisimple representations are semisimple.

Proposition 1.4. Let $V_1, V_2, \cdots, V_m$ be a finite list of irreducible finite-dimensional A-representations with $V_i \not\cong V_j$ if $i \neq j$. Consider the A-representation $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$ where $n_1, n_2, \ldots, n_m$ are nonnegative integers. Now suppose W is a subrepresentation of V. Then:

1. For some integers $0 \leq r_i \leq n_i$ there is an isomorphism $\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$.
2. The map $\bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\phi} W \hookrightarrow V$ is a direct sum of inclusions $\phi_i : V_i^{\oplus r_i} \hookrightarrow V_i^{\oplus n_i}$ of the form
$$\phi_i(a_1, a_2, \cdots, a_{r_i}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_{r_i} \end{bmatrix} X_i$$
where each X_i is a full rank $r_i \times n_i$ matrix with values in K.

Remark 1.5. Suppose $V_1, V_2, \cdots, V_m$ are irreducible, pairwise non-isomorphic, finite-dimensional A-representations. Choose positive integers $n_1, n_2, \ldots, n_m$ and define $V = \bigoplus_{i=1}^m V_i^{\oplus n_i}$. Then any subrepresentation W of V has $W \cong \bigoplus_{i=1}^m V_i^{\oplus r_i}$ for some integers $0 \leq r_i \leq n_i$, and there is an isomorphism
$$\phi : \bigoplus_{i=1}^m V_i^{\oplus r_i} \xrightarrow{\sim} W$$
that sends $x = \begin{bmatrix} \underbrace{x_{11}\, x_{12} \cdots x_{1r_1}}_{\in V_1^{\oplus r_1}} \; \underbrace{x_{21}\, x_{22} \cdots x_{2r_2}}_{\in V_2^{\oplus r_2}} \cdots \; \underbrace{x_{m1}\, x_{m2} \cdots x_{mr_m}}_{\in V_m^{\oplus r_m}} \end{bmatrix} \in \bigoplus_{i=1}^m V_i^{\oplus r_i}$ to $xM \in W$, where M is a full rank, block diagonal matrix with entries in K, whose successive blocks have size $r_i \times n_i$ for $i = 1, 2, \ldots, m$.

Proof. sketch: If $W = 0$ then the proposition is trivial. Assume $W \neq 0$.

We proceed by induction on $n \overset{\text{def}}{=} n_1 + n_2 + \cdots + n_m$.

If $n = 1$ then we must have $0 \neq W = V_i$ in which case the result is again obvious.

Assume $n > 1$. Since W is finite-dimensional, it has an irreducible subrepresentation P (this was shown in the HW1 exercises). Observe that $\mathrm{Hom}_A(P, V) = \bigoplus_{i=1}^m \mathrm{Hom}_A(P, V_i)^{\oplus n_i}$. In this equation:

- each term $\mathrm{Hom}_A(P, V_i)$ on the right side is nonzero if and only if $P \cong V_i$ by Schur's lemma;
- the left side is nonzero since it contains inclusion $P \hookrightarrow W \hookrightarrow V$.

Therefore P must be isomorphic to V_i for some i.

The inclusion $V_i \xrightarrow{\sim} P \hookrightarrow V_i^{\oplus n_i} \hookrightarrow V$ must be given by a map of the form
$$v \mapsto (q_1 v, \cdots, q_{n_i} v)$$
for some scalars $q_i \in K$ that are not all zero. This is because composing this map with each projection
$$(a_1, \cdots, a_{n_i}) \mapsto a_j \in V_i$$
is a morphism of A-representations $V_i \to V_i$, which must be a scalar map by Schur's lemma.

Let $g \in \mathrm{GL}_{n_i}(K) = \{\text{invertible } n_i \times n_i \text{ matrices}\}$ act on $V_i^{\oplus n_i}$ on the right by the formula

$$g : (v_1, v_2, \cdots, v_{n_i}) \mapsto \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} g$$

while acting on $V_j^{\oplus n_j}$ for $i \neq j$ as the identity. This gives a right action of the general linear group on V.
We may choose $g \in \mathrm{GL}_{n_i}(K)$ such that

$$Pg = \{(0, 0, \cdots, 0, v) : v \in V_i\} \subset V_i^{\oplus n_i}.$$

Then $Wg = W' \oplus V_i$ where $V_i = Pg$ and W' is the kernel of projection $Wg \to Pg$, which satisfies

$$W' \subset V_1^{\oplus n_1} \oplus \cdots \oplus V_i^{\oplus (n_i - 1)} \oplus \cdots \oplus V_m^{\oplus n_m}.$$

Now we apply the proposition to W' by induction, and multiply the resulting inclusion by g^{-1}. □

Corollary 1.6. Assume the following setup:

- V is an irreducible finite-dimensional representation of A.
- The elements $v_1, v_2, \ldots, v_n \in V$ are linearly independent.
- The elements $w_1, w_2, \ldots, w_n \in V$ are arbitrary.

Then there exists an element $a \in A$ such that $av_i = w_i$ for all $i = 1, 2, \ldots, n$.

Proof. Assume no such element exists. Then the image of A under the map

$$a \mapsto (av_1, \cdots, av_n)$$

is a proper subrepresentation of $V^{\oplus n}$, which we denote by W.
By Proposition 1.4 we know that $W \cong V^{\oplus m}$ for some $0 \leq m < n$ and there exists an inclusion

$$\phi : V^{\oplus m} \xrightarrow{\sim} W \hookrightarrow V^{\oplus n}$$

of the form $\phi(a_1, a_2, \cdots, a_m) = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X$ where X is a full rank $m \times n$ matrix.
Since $(v_1, v_2, \cdots, v_n) \in W$, we may choose $a_i \in V$ such that $\phi(a_1, a_2, \cdots, a_m) = (v_1, v_2, \cdots, v_n)$.
Also, since $m < n$, there is nonzero vector $\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \in K^n$ such that $X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$. But now

$$\sum_{i=1}^{n} q_i v_i = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$$

which contradicts the linear independence of $v_1, \cdots, v_n$. □

Theorem 1.7 (Density theorem)**.** Let (V, ρ) be an irreducible, finite-dimensional A-representation. Then the map $\rho : A \to \mathrm{End}(V)$ is surjective. Moreover, if $(V, \rho) = (V_1, \rho_1) \oplus \cdots \oplus (V_r, \rho_r)$ where each (V_i, ρ_i) is an irreducible A-representation, then the map $\bigoplus_{i=1}^{r} \rho_i : A \to \bigoplus_{i=1}^{r} \mathrm{End}(V_i)$ is also surjective.

Proof. For the first claim, choose any $L \in \mathrm{End}(V)$ and suppose $v_1, v_2, \ldots, v_n$ is a basis of V. Set $w_i = L(v_i)$. By the previous corollary, some $a \in A$ has $\rho(a)v_i = w_i$ for all i which means that $\rho(a) = L$.

The second claim is nontrivial since direct sums of surjective maps are not necessarily surjective. For example, the direct sum of the identity map becomes $x \mapsto (x, x, \ldots, x)$ which is usually not surjective.

The surjective property that we wish to show will be a consequence of the second part of Proposition 1.4.

Let $Y = \bigoplus_{i=1}^r \mathrm{End}(V_i)$. This is a semisimple A-representation as $\mathrm{End}(V_i) \cong V_i^{\oplus d_i}$ where $d_i = \dim V_i$.

By Proposition 1.4, the subrepresentation

$$W \overset{\text{def}}{=} \left(\bigoplus_{i=1}^r \rho_i \right)(A) \subset Y$$

is isomorphic to $\bigoplus_{i=1}^r V_i^{\oplus m_i}$ for some integers $0 \le m_i \le d_i$, and there is an inclusion

$$\phi : \bigoplus_{i=1}^r V_i^{\oplus m_i} \xrightarrow{\sim} W \hookrightarrow Y$$

that is given by a direct sum of inclusions $\phi_i : V_i^{\oplus m_i} \hookrightarrow V_i^{\oplus d_i}$.

Since each ρ_i is surjective, the composition of this inclusion with the projection $Y \to \mathrm{End}(V_i)$ is surjective. Hence each ϕ_i is surjective and $m_i = d_i$. This shows that $\bigoplus_i \rho_i$ is surjective. $\square$

2 matrix algebras, filtrations, finite-dimensional algebras

2.1 Matrix algebras

We have already seen that the algebra of all $n \times n$ matrices over K has a unique isomorphism class of irreducible representations. We can generalize this to block diagonal matrix algebras.

Choose integers $d_1, d_2, \ldots, d_r > 0$.

Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(K)$ where we define $\mathrm{Mat}_d(K)$ to be the algebra of $d \times d$ matrices over K. Set $n = \sum_{i=1}^r d_i$. Then we can view A as the subalgebra of $\mathrm{Mat}_n(K)$ consisting of all block diagonal matrices with successive blocks of size $d_i \times d_i$.

The vector space K^n is automatically an A-representation. We construct a sequence of sub-representations:

Let $V_1 \subseteq K^n$ be the subspace of vectors with zeros outside rows $1, 2, \ldots, d_1$

Let $V_2 \subset K^n$ be the subspace of vectors with zeros outside rows $d_1 + 1, d_1 + 2, \ldots, d_1 + d_2$.

Let $V_3 \subset K^n$ be the subspace of vectors with zeros outside rows $d_1 + d_2 + 1, d_1 + d_2 + 2, \ldots, d_1 + d_2 + d_3$.

Define $V_4, \ldots, V_r$ analogously, so $V_r \subseteq K^n$ is the subspace of vectors with zeros outside the last d_r rows.

As vector spaces, we have $V_i \cong K^{d_i}$.

Theorem 2.1. In this setup, each V_i is an irreducible A-representation, and every finite-dimensional A-representation is isomorphic to a direct sum of zero or more copies of $V_1, V_2, \ldots, V_r$.

Before proving this theorem, we introduce another definition.

Definition 2.2 (dual representation)**.** Suppose (V, ρ) is an A-representation. Let V^* be the vector space of all K-linear maps $\lambda : V \to K$. Then let $\rho^* : A \to \mathrm{End}(V^*)$ be the linear map defined by

$$\rho^*(a)(\lambda) : x \mapsto \lambda(\rho(a)(x)) \quad \text{for } a \in A \text{ and } \lambda \in V^*.$$

We refer to the pair (V^*, ρ^*) as the *dual* of (V, ρ).

It is a representation of the opposite algebra A^{op}.

Fact 2.3. For $A = \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(K) \subseteq \mathrm{Mat}_n(K)$, the usual matrix transpose map $X \mapsto X^\top$ is an algebra isomorphism $A \cong A^{\mathrm{op}}$.

Given a linear map between vector spaces $L : V \to W$, define $L^* : W^* \to V^*$ by $L^*(f) = f \circ L$.

Fact 2.4. If L is injective then L^* is surjective, and if L is surjective then L^* is injective.

Let's proof the theorem 2.1

Proof. It is easy to see that each V_i is an irreducible A-representation, as each nonzero element of V_i is cyclic for the action of A.

Let X be some finite m-dimensional representation of A where $m < \infty$.

Then X^* is representation of $A^{\mathrm{op}} \cong A$.

In other words, X^* can be viewed as an A-representation for the action

$$a \cdot \lambda : x \mapsto \lambda(a^\top x) \quad \text{for } x \in X,\ \lambda \in X^*,\ a \in A.$$

Choose a basis $\{\lambda_1, \ldots, \lambda_m\}$ for X^*. Then let $\phi : A \oplus \cdots \oplus A = A^{\oplus m} \to X^*$ be the map

$$\phi(a_1, a_2, \ldots, a_m) = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_m\lambda_m.$$

Because $K \subset A$, this map is surjective. Therefore, the dual map $\phi^* : X \to (A^{\oplus m})^*$ is injective.

Key claim: The A-representations $(A^{\oplus m})^*$ and $A^{\oplus m}$ are isomorphic.

If we can prove this, then it will follow that X is isomorphic to a subrepresentation of $A^{\oplus m}$. As we have $A \cong \bigoplus_{i=1}^{r} V_i^{\oplus d_i}$ as A-representations (the isomorphism is provided by viewing a matrix as a tuple of column vectors), we would then get

$$X \cong \left(\text{a subrepresentation of } A^{\oplus m} \cong \bigoplus_{i=1}^{r} V_i^{\oplus m d_i} \right),$$

which by our technical proposition would imply that $X \cong \bigoplus_{i=1}^{r} V_i^{\oplus s_i}$ for some integers $s_i \geq 0$ as desired.

We will only explain the $m = 1$ case of the key claim.

Let A act on A^* by $a \cdot \lambda : x \mapsto \lambda(a^\top x)$ for $a \in A$ and $\lambda \in A^*$. Define $\Theta : A \to A^*$ to be the linear map

$$\Theta : a \mapsto (x \mapsto \mathrm{tr}(a^\top x)).$$

Then Θ is a bijection since it is a nonzero linear map with trivial kernel between finite-dimensional vector spaces of the same dimension. It is also a homomorphism of A-representations since we have

$$\Theta(gh)(x) = \mathrm{tr}(h^\top g^\top x) = \Theta(h)(g^\top x) = (g \cdot \Theta(h))(x) \quad \text{for } g, h, x \in A,$$

which implies that $\Theta(gh) = g \cdot \Theta(h)$. Thus $\Theta : A \xrightarrow{\sim} A^*$ is an isomorphism of A-representations. □

2.2 Filtrations

Continue to let A be an algebra. Suppose V is an A-representation.

Definition 2.5 (filtration)**.** A **filtration** of V is a finite, increasing sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where each V_i is sub-representation of V.

Lemma 2.6. If $\dim V < \infty$ then V has a filtration in which each quotient V_i/V_{i-1} is an irreducible A-representation.

Proof. We argue by induction on $\dim V$.

If $\dim V \leq 1$ then the result is trivial: just take $n = 1$ and $V_n = V$.

Assume $\dim V > 1$ and choose any irreducible subrepresentation $V_1 \subset V$.

Then let $U = V/V_1$. By induction there is a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} = U$$

in which each quotient U_i/U_{i-1} is irreducible.

Let V_i be the preimage of U_{i-1} under the quotient map $V \to V/V_1 = U$. Then

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

gives the desired filtration, since $V_i/V_{i-1} \cong (V_i/V_1)/(V_{i-1}/V_1) = U_{i-1}/U_{i-2}$ for $i > 1$. □

2.3 Radicals of finite-dimensional algebras

Assume that A is an algebra with $\dim A < \infty$.

Definition 2.7 (radical)**.** The **radical** of A is the set of elements $a \in A$ that act as zero in every irreducible representation of A. Let $\mathrm{Rad}(A)$ denote this set of elements.

Proposition 2.8. The set $\mathrm{Rad}(A)$ is a two-sided ideal of A.

Proof. The set $\mathrm{Rad}(A)$ is a subspace of A since if (V, ρ) is a representation then

$$\rho(x) = 0 \implies \rho(cx) = c\rho(x) = 0 \quad \text{and} \quad \rho(x) = 0 = \rho(y) \implies \rho(x+y) = \rho(x) + \rho(y) = 0$$

for all $x, y \in A$ and $c \in K$. It is also a two-sided ideal since if $a, b \in A$ then

$$\rho(x) = 0 \implies \rho(axb) = \rho(a)\rho(x)\rho(b) = 0.$$

□

Let I be a two-sided ideal in A. For integers $n \geq 1$, let $I^n = K\text{-span}\{x_1 x_2 \cdots x_n : x_i \in I\}$.

We say that I is *nilpotent* if $I^n = 0$ for some $n > 0$.

For example, the subspace of strictly upper triangular matrices is a nilpotent ideal in the algebra of all upper triangular $n \times n$ matrices over K.

Proposition 2.9. If I is a nilpotent two-sided ideal in A then $I \subseteq \mathrm{Rad}(A)$.

Proof. Suppose I is a nilpotent two-sided ideal with $I^n = 0$. Choose any irreducible A-representation V and pick $0 \neq v \in V$. Then the subspace $Iv \overset{\text{def}}{=} \{xv : x \in I\}$ is a subrepresentation. If $Iv = V$ then there is some $x \in I$ with $xv = v$, which is impossible as $x^n = 0$. Therefore $Iv = 0$ as it is a proper subrepresentation of an irreducible representation. Since V was arbitrary, it follows that $I \subseteq \mathrm{Rad}(A)$. □

The following shows that $\mathrm{Rad}(A)$ is precisely the largest nilpotent two-sided ideal in A.

Proposition 2.10. $\mathrm{Rad}(A)$ is a nilpotent two-sided ideal.

Proof. Since $\dim A < \infty$, the previous section shows that there exists a filtration of the regular representation $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ in which each quotient A_i/A_{i-1} is irreducible as an A-representation.

Each $x \in \mathrm{Rad}(A)$ acts as zero on A_i/A_{i-1}, which means that $xA_i \subset A_{i-1}$.

Therefore if $x_1, x_2, \cdots \in \mathrm{Rad}(A)$ then $x_1x_2\cdots x_iA_i \subset A_0 \subset A_1 \subset \cdots \subset A_{i-1}$ and $x_1x_2\cdots x_nA_n = 0$. Hence $\mathrm{Rad}(A)^n = 0$. □

2.4 Representations of nite-dimensional algebras

As a final application today, we can "classify" all representations of finite-dimensional algebras.

Theorem 2.11. Suppose A is a finite-dimensional algebra. Then A has finitely many isomorphism classes of irreducible representations $V_1, V_2, \ldots, V_r$ and $A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^r \mathrm{End}(V_i)$ as K-algebras. Moreover, every irreducible A-representation is finite-dimensional.

Notice that since $\dim V_i$ is finite, we have $\mathrm{End}(V_i) \cong \mathrm{Mat}_d(K)$ for $d = \dim V_i$.

Therefore $A/\mathrm{Rad}(A)$ is isomorphic to a block diagonal matrix algebra of the form considered earlier today.

Proof. Suppose V is an A-representation.

If $0 \neq x \in V$ then Ax is a nonzero subrepresentation of dimension at most $\dim A < \infty$.

Therefore, if V is irreducible then we must have $V = Ax$ and $\dim V \leq \dim A < \infty$.

Now suppose $(V_1, \rho_1), \ldots, (V_r, \rho_r)$ are pairwise non-isomorphic, irreducible A-representations.

By the *density theorem*, the direct sum

$$\phi = \bigoplus_{i=1}^r \rho_i : A \to \bigoplus_{i=1}^r \mathrm{End}(V_i)$$

is a surjective map. Since each $\mathrm{End}(V_i)$ has dimension $(\dim V_i)^2$, we have

$$r \leq \sum_{i=1}^r (\dim V_i)^2 \leq \dim A < \infty$$

Thus r cannot be arbitrarily large, so the number of distinct isomorphism classes of irreducible A-representations is finite and at most $\dim A$.

Finally assume r is maximal above, so that every irreducible A-representation is isomorphic to some V_i.

Then $\mathrm{Rad}(A) = \ker(\phi)$ so ϕ passes to an isomorphism $A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^r \mathrm{End}(V_i)$. □

Corollary 2.12. If $V_1, V_2, \ldots, V_r$ are pairwise non-isomorphic irreducible representations of a finite-dimensional algebra A then $\sum_{i=1}^r (\dim V_i)^2 \leq \dim A$.

3 semisimple algebras, characters, two general theorems

3.1 Semisimple algebras

Our main new results today concern the following class of algebras.

Definition 3.1 (semisimple algebra)**.** A finite-dimensional algebra A is called **semisimple** if $\mathrm{Rad}(A) = 0$.

Recall that a representation is **semisimple** if it is a direct sum of irreducible representations.

Theorem 3.2. Suppose $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\mathbb{K})$ for some integers $d_1, d_2, \dots, d_r > 0$. For each index $i \in \{1, 2, \dots, r\}$, A has an irreducible representation V_i of dimension d_i, and every finite-dimensional representation of A is a direct sum of copies of $V_1, V_2, \dots, V_r$, which are pairwise non-isomorphic.

If we view $A \subseteq \mathrm{Mat}_n(\mathbb{K})$ as a subalgebra of block diagonal $n \times n$ matrices where $n = d_1 + d_2 + \cdots + d_r$, then we can construct V_i as the subspace of vectors in $\mathbb{K}^n$ with zeros outside the rows indexed by

$$(d_1 + d_2 + \cdots + d_{i-1}) + \{1, 2, \dots, d_i\}.$$

Theorem 3.3. A finite-dimensional algebra A has finitely many irreducible representations $V_1, \dots, V_r$ up to isomorphism, each representation V_i has finite dimension $d_i = \dim(V_i)$, and it holds that

$$A/\mathrm{Rad}(A) \cong \bigoplus_{i=1}^r \mathrm{End}(V_i) \cong \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\mathbb{K}).$$

Proposition 3.4. Assume A is an algebra over $\mathbb{K}$ with $\dim A < \infty$. The following are equivalent:

1. A is semisimple.
2. $\sum_{i=1}^r \dim(V_i)^2 = \dim A$ where V_i are the distinct isomorphism classes of irreducible A-representations.
3. $A \cong \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\mathbb{K})$ for some integers $d_1, d_2, \dots, d_r > 0$
4. Any finite-dimensional representation of A is semisimple.
5. The regular representation of A is semisimple.

Proof. We have (1) $\iff$ (2) since $\dim A - \dim \mathrm{Rad}(A) = \sum_{i=1}^r \dim(V_i)^2$.

The implication (1) $\implies$ (3) is Theorem 3.3. Conversely, (3) + Theorem 3.2 $\implies$ (2) $\implies$ (1). We conclude that (1) $\iff$ (3).

Now we claim that (3) $\implies$ (4) $\implies$ (5) $\implies$ (3).

The implication (3) $\implies$ (4) holds by Theorem 3.2 and (4) $\implies$ (5) is trivial.

To show that (5) $\implies$ (3), assume (5). Then we can write $A = \bigoplus_{i=1}^r d_i V_i$ where $V_1, V_2, \dots, V_r$ are irreducible and pairwise-non-isomorphic, since the regular representation of A is semisimple.

Now consider $\mathrm{End}_A(A) = \{\text{morphisms } A \to A \text{ as } A\text{-representations}\} = \mathrm{Hom}_A(A, A)$.

Schur's lemma tells us that

- $\mathrm{End}_A(V_i) = \mathbb{K}$ so $\mathrm{End}_A(d_i V_i) \cong \mathrm{Mat}_{d_i}(\mathbb{K})$, and
- $\mathrm{Hom}_A(V_i, V_j) = 0$ if $i \neq j$, so $\mathrm{Hom}_A(d_i V_i, d_j V_j) = 0$ if $i \neq j$.

Thus, we compute $\mathrm{End}_A(A) = \mathrm{Hom}_A(A, A) = \bigoplus_{i,j} \mathrm{Hom}(d_i V_i, d_j V_j) \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$.

Exercise: Show that $(\mathrm{End}_A(A))^{\mathrm{op}} \cong A$ or equivalently that $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$.

Last time: There is an isomorphism $(\bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K}))^{\mathrm{op}} \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$ afforded by the transpose map.

Thus we have $A \cong (\mathrm{End}_A(A))^{\mathrm{op}} \cong (\bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K}))^{\mathrm{op}} \cong \bigoplus_i \mathrm{Mat}_{d_i}(\mathbb{K})$.

This is property (3), so (5) $\implies$ (3) as desired. □

3.2 Characters

Let A be an algebra. Suppose (V, ρ) is a finite-dimensional representation of A.

Definition 3.5 (character)**.** The **character** of (V, ρ) is the linear map $\chi_{(V,\rho)} : A \to \mathbb{K}$ with the formula

$$\chi_{(V,\rho)}(a) = \mathrm{tr}(\rho(a)) \quad \text{for } a \in A.$$

How can we compute the trace of $\phi \in \mathrm{End}(V)$?

First choose a basis $e_1, e_2, \ldots, e_n$ of V. Then $\mathrm{tr}(\phi) = \sum_{i=1}^n$ (coefficient of e_i in $\phi(e_i)$).

Some basic and well-known facts about traces:

1. The method just given to compute the trace does not depend on the choice of basis.
2. We have $\mathrm{tr}(\phi_1\phi_2) = \mathrm{tr}(\phi_2\phi_1)$ for all $\phi_1, \phi_2 \in \mathrm{End}(V)$, so $\mathrm{tr}(\phi_1\phi_2\phi_1^{-1}) = \mathrm{tr}(\phi_2)$ if ϕ_1 is invertible.
3. If $(V_1, \rho_1) \cong (V_2, \rho_2)$ are finite-dimensional A-representations then $\chi_{(V_1,\rho_1)} = \chi_{(V_2,\rho_2)}$.

To abbreviate, we will sometimes write χ_V instead of $\chi_{(V,\rho)}$.

Let $[A, A] = \mathbb{K}\text{-span}\left\{[a, b] \overset{\text{def}}{=} ab - ba : a, b \in A\right\}$. We view this as just a vector space.

Fact 3.6. We always have $[A, A] \subseteq \ker(\chi_{(V,\rho)})$

Proof. Let $\chi = \chi_{(V,\rho)}$. Then $\chi(ab - ba) = \mathrm{tr}(\rho(ab)) - \mathrm{tr}(\rho(ba)) = \mathrm{tr}(\rho(a)\rho(b)) - \mathrm{tr}(\rho(b)\rho(a)) = 0$ □

In the following theorem, $\dim A$ is not required to be finite.

Theorem 3.7. The characters of any list of non-isomorphic irreducible finite-dimensional A-representations are linearly independent (and, in particular, are distinct).

Proof. Suppose $(V_1, \rho_1), (V_2, \rho_2), \ldots, (V_r, \rho_r)$ are pairwise non-isomorphic irreducible finite-dimensional A-representations. Let $\chi_i = \chi_{(V_i,\rho_i)}$. By the density theorem, the map

$$\rho_1 \oplus \cdots \oplus \rho_r : A \to \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_r)$$

is surjective. Therefore, if $\sum_{i=1}^r \lambda_i \chi_i(a) = 0$ for all $a \in A$ for some coefficients $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{K}$, then

$$\sum_{i=1}^r \lambda_i \mathrm{tr}(M_i) = 0 \quad \text{for any } M_i \in \mathrm{End}(V_i) \text{ chosen independently,}$$

which is only possible if $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. □

We say that a character $\chi_{(V,\rho)}$ is **irreducible** if (V, ρ) is irreducible.

Theorem 3.8. Assume A is semisimple and $\dim A < \infty$. Then the irreducible characters of A are a basis for the vector space $(A/[A, A])^*$ of linear maps $A/[A, A] \to \mathbb{K}$.

Proof. Each character χ has $[A, A] \subset \ker(\chi)$, so χ belongs to $(A/[A, A])^*$.

Since $A = \mathrm{Mat}_{d_1}(\mathbb{K}) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(\mathbb{K})$ it follows that $[A, A] = \bigoplus_{i=1}^r [\mathrm{Mat}_{d_i}(\mathbb{K}), \mathrm{Mat}_{d_i}(\mathbb{K})]$.

We claim that $[\mathrm{Mat}_d(\mathbb{K}), \mathrm{Mat}_d(\mathbb{K})] = \mathfrak{sl}_d(\mathbb{K})$, where $\mathfrak{sl}_d(\mathbb{K})$ is the vector space of $d \times d$ matrices over $\mathbb{K}$ with zero trace. To prove the claim, note that the trace map certainly vanishes on $[\mathrm{Mat}_d(\mathbb{K}), \mathrm{Mat}_d(\mathbb{K})]$ and that $\mathfrak{sl}_d(\mathbb{K})$ is spanned by the commutators

$$E_{ij} = [E_{ik}, E_{kj}] \text{ for } i \neq j \quad \text{and} \quad E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}].$$

where E_{ij} is the elementary matrix with 1 in entry (i, j) and 0 elsewhere.

With the claim proved, we have $A/[A, A] \cong \mathbb{K}^r$ since $\mathrm{Mat}_d(\mathbb{K})/\mathfrak{sl}_d(\mathbb{K}) \cong \mathbb{K}$.

Finally, we know that A has r distinct irreducible characters (by Theorem 3.2), and these are linearly independent elements of $(A/[A, A])^*$, so they must be a basis as $\dim(A/[A, A])^* = \dim(A/[A, A]) = r$. □

Two general results

We finish today with two general results that can be applied to algebras A that are not necessarily semisimple. Assume $\dim A < \infty$. Let V be a finite-dimensional representation of A.

Theorem 3.9 (Jordan-Hölder theorem)**.** Suppose we have filtrations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \qquad \text{and} \qquad 0 = V'_0 \subset V'_1 \subset \cdots \subset V'_m = V$$

where V_i and V'_i are subrepresentations such that the quotients $W_i := V_i/V_{i-1}$ and $W'_i := V'_i/V'_{i-1}$ are irreducible. Then $n = m$ and there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $W_{\sigma(i)} \cong W'_i$ for all i.

We call the common length $m = n$ of these filtrations the *length* of the representation V.

Proof. We can give a simple proof when $\mathrm{char}(\mathbb{K}) = 0$. In this case, it follows by the additive property over exact sequences that $\chi_V = \chi_W + \chi_{V/W}$ if W is any subrepresentation of V, and so we have $\chi_V = \sum_{i=1}^n \chi_{W_i} = \sum_{i=1}^m \chi_{W'_i}$.

Then we can deduce the theorem by the linear independence of the irreducible characters of A.

This argument does not work for $\mathrm{char}(\mathbb{K}) = p > 0$, because the multiplicities of the irreducible characters in the decomposition of χ_V could be multiples of p. One can handle this case by a more involved inductive argument; see the textbook for the details. □

We maintain the same setup for A and V in the next theorem.

Theorem 3.10 (Krull-Schmidt theorem)**.** There is a decomposition of V, which is unique up to isomorphism and rearrangement of factors, as a direct sum of indecomposable A-representations.

We will give the proof next time. While the existence of such a decomposition follows pretty easily by induction on $\dim V$, the uniqueness claim in the theorem is nontrivial.

4 Krull-Schmidt theorem, tensor products of algebras

4.1 Two general theorems

Our goal today is to establish two general theorems about representations of an algebra A that is not necessarily semisimple. We proved the first of these theorems last time:

Theorem 4.1 (Jordan-Hölder theorem)**.** If V is an A-representation with $\dim V < \infty$ then there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where each V_i is a subrepresentation and each quotient V_i/V_{i-1} is irreducible. Moreover, any other filtration with these properties has same length n and the same irreducible quotients up to isomorphism and permutations of indices.

Today we will supply the proof of the next theorem:

Theorem 4.2 (Krull-Schmidt theorem)**.** If V is an A-representation with $\dim V < \infty$ then there exists a decomposition $V = \bigoplus_{i\in I} V_i$ where each V_i is an indecomposable subrepresentation, and this decomposition is unique up to isomorphism and rearrangement of factors (permutation).

Remember that when A is semisimple, every indecomposable representation is irreducible, but for a general algebra we may not be able to decompose a representation into a direct sum of irreducible subrepresentations. The Krull-Schmidt theorem is relevant to the latter setting.

We will prove the Krull-Schmidt theorem after establishing a few lemmas.

A linear map $\theta : W \to W$ is **nilpotent** if $\theta^N := \theta \circ \theta \circ \cdots \circ \theta$ is zero for some $N > 0$.

Lemma 4.3. Let W be an indecomposable A-representation where $\dim W < \infty$. Suppose $\theta : W \to W$ is a morphism of A-representations. Then θ is either an isomorphism or nilpotent.

Proof. For $\lambda \in \mathbb{K}$, the *generalized λ-eigenspace* of θ is

$$W_\lambda := \{x \in W : (\theta - \lambda)^N(x) = 0 \text{ for some } N > 0\}.$$

The subspace W_λ is nonzero if and only if λ is an eigenvalue of θ.

By standard linear algebra over algebraically closed fields, we know that $W = \bigoplus_\lambda W_\lambda$ where the direct sum is over the eigenvalues of θ. Observe that each W_λ is an A-subrepresentation.

Since W is indecomposable, θ must only have one eigenvalue λ. If $\lambda = 0$ then θ is nilpotent since $W = W_0$.

If $\lambda \neq 0$ then θ is invertible, and hence an isomorphism of A-representations. $\square$

Lemma 4.4. Let W be an indecomposable A-representation where $\dim W < \infty$. Suppose $\theta_s : W \to W$ for $s = 1, 2, \ldots, n$ are nilpotent morphisms of A-representations. Then $\theta := \theta_1 + \cdots + \theta_n$ is also nilpotent.

Proof. We argue by contradiction. Let n be minimal such that the lemma fails.

Then we must have $n > 1$ and θ is not nilpotent. Hence θ is invertible by previous lemma.

Therefore we can write $1 = \theta^{-1}\theta = \sum_{s=1}^n \theta^{-1}\theta_s$.

Since $\ker(\theta^{-1}\theta_s) = \theta^{-1}(\ker(\theta_s)) \neq 0$, each $\theta^{-1}\theta_s$ is not invertible and therefore nilpotent by the lemma.

But then $1 - \theta^{-1}\theta_n = \sum_{s=1}^{n-1} \theta^{-1}\theta_s$ is invertible, and therefore not nilpotent, since if X is nilpotent then

$$(1 - X)^{-1} = 1 + X + X^2 + \cdots.$$

This contradicts the minimality of n, so we conclude that the lemma actually holds for all n. $\square$

We now return to the proof of the Krull-Schmidt theorem 4.2.

Proof. To show the existence of an indecomposable decomposition $V = \bigoplus_{i\in I} V_i$, note that if V is not indecomposable then must exist nonzero subrepresentations U and W with $V = U \oplus W$, and by induction on dimension we can assume that U and W already have indecomposable decompositions.

The hard part is showing the uniqueness of the resulting decomposition.

Suppose $V = \bigoplus_{s=1}^m V_s = \bigoplus_{s=1}^n W_s$ where each V_s and W_s is an indecomposable subrepresentation. Let

$$\begin{array}{ll} i_s : V_s \hookrightarrow V & p_s : V \twoheadrightarrow V_s \\ j_s : W_s \hookrightarrow V & q_s : V \twoheadrightarrow W_s \end{array}$$

be the natural inclusion and projection maps.

Define $\theta_s = p_1 \circ j_s \circ q_s \circ i_1$ so that

$$\theta_s : V_1 \xrightarrow{i_1} V \xrightarrow{q_s} W_s \xrightarrow{j_s} V \xrightarrow{p_1} V_1.$$

Note that i_s, p_s, j_s, q_s, and θ_s are all morphisms of A-representations.

Also, notice that the sum $\theta_1 + \theta_2 + \cdots + \theta_n$ is the identity map $V_1 \to V_1$.

Each θ_s is either nilpotent or an isomorphism by Lemma 4.3.

Since $\sum_{s=1}^n \theta_s$ is not nilpotent, some θ_s is an isomorphism by Lemma 4.4.

Without loss of generality we can assume that $\theta_1 : V_1 \to V_1$ is an isomorphism. Since

$$\theta_1 : V_1 \xrightarrow{q_1\circ i_1} W_1 \xrightarrow{p_1\circ j_1} V_1$$

is an isomorphism, we must have $W_1 = \mathrm{image}(q_1 \circ i_1) \oplus \ker(p_1 \circ j_1)$.

As W_1 is indecomposable, both $p_1 \circ j_1 : W_1 \to V_1$ and $q_1 \circ i_1 : V_1 \to W_1$ must be isomorphisms.

Let $V' = \bigoplus_{s=2}^m V_s$ and $W' = \bigoplus_{s=2}^n W_s$ so that $V = V_1 \oplus V' = W_1 \oplus W'$. Let

$$h : V' \hookrightarrow V \twoheadrightarrow W'$$

be the composition of the obvious inclusion and projection maps.

Clearly $\ker(h) = V' \cap W_1$, but $(p_1 \circ j_1)(V' \cap W_1) = 0$.

Since $p_1 \circ j_1 : W_1 \to V_1$ is isomorphism, must have $\ker(h) = 0$ so $h : V' \to W'$ is isomorphism.

Now by induction applied to the decompositions

$$V' = \bigoplus_{s=2}^m V_s \cong \bigoplus_{s=2}^n W_s = W', \tag{1}$$

we must have $m = n$ and there must exist a permutation σ with $V_s \cong W_{\sigma(s)}$ for all s.

This establishes that the same holds for our starting decompositions $V = \bigoplus_{s=1}^m V_s = \bigoplus_{s=1}^n W_s$. $\square$

4.2 Tensor products of algebras and representations

To finish today's lecture, we briefly discuss representations of tensor product algebras.

Let A and B be $\mathbb{K}$-algebras and write $\otimes = \otimes_{\mathbb{K}}$ for the usual tensor product for $\mathbb{K}$-vector spaces.

Since A and B are vector spaces, we can consider the vector space $A \otimes B$. It has more structure:

Fact 4.5. The vector space $A \otimes B$ is itself a $\mathbb{K}$-algebra for the product given by the bilinear operation

$$(A \otimes B) \times (A \otimes B) \to A \otimes B$$

satisfying $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ for $a, a' \in A, b, b' \in B$. The unit for this product is $1_A \otimes 1_B$.

Let V be an A-representation and let W be a B-representation. Then $V \otimes W$ has a unique structure as an $A \otimes B$-representation in which $(a \otimes b)(v \otimes w) := av \otimes bw$ for $a \in A, b \in B, v \in V, w \in W$.

Theorem 4.6. Assume $\dim V < \infty$ and $\dim W < \infty$. Then $V \otimes W$ is irreducible (as an $A \otimes B$-representation) if V and W are irreducible (as A- and B-representations).

Proof. Assume V and W are both irreducible and of finite dimension.

By the density theorem, we have surjective maps $\rho_V : A \to \mathrm{End}(V)$ and $\rho_W : A \to \mathrm{End}(W)$.

Check that $\rho_V \otimes \rho_W : A \otimes B \to \mathrm{End}(V) \otimes \mathrm{End}(W)$ is also surjective.

If $\dim V < \infty$ and $\dim W < \infty$ then there is an isomorphism $\mathrm{End}(V) \otimes \mathrm{End}(W) \cong \mathrm{End}(V \otimes W)$.

But the map $\rho_{V\otimes W} : A \otimes B \to \mathrm{End}(V \otimes W)$ is thus surjective as it is the composition

$$A \otimes B \xrightarrow{\rho_V \otimes \rho_W} \mathrm{End}(V) \otimes \mathrm{End}(W) \xrightarrow{\cong} \mathrm{End}(V \otimes W).$$

Hence $V \otimes W$ is irreducible, since $\rho_{V\otimes W}$ being surjective implies that every $0 \neq x \in V \otimes W$ is cyclic. $\square$

The previous theorem has a converse.

Theorem 4.7. Suppose M is an irreducible $A \otimes B$-representation of finite dimension. Then $M \cong V \otimes W$ for some irreducible A-representation V and irreducible B-representation W.

Proof. **Sketch**: We can assume A and B are finite-dimensional by replacing each algebra by its image under

$$A \hookrightarrow A \otimes B \twoheadrightarrow \mathrm{End}(M) \qquad \text{and} \qquad B \hookrightarrow A \otimes B \twoheadrightarrow \mathrm{End}(M)$$

where the inclusions send $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$. Next, check that

$$\mathrm{Rad}(A \otimes B) = \mathrm{Rad}(A) \otimes B + A \otimes \mathrm{Rad}(B)$$

so $(A \otimes B)/\operatorname{Rad}(A \otimes B) \cong A/\operatorname{Rad}(A) \otimes B/\operatorname{Rad}(B)$ and M is an irreducible representation of this quotient.

Finally, the result can be deduced by identifying the quotient algebras $A/\operatorname{Rad}(A)$ and $B/\operatorname{Rad}(B)$ with explicit (direct sums of) matrix algebras, using the classification of irreducible representations for such algebras and the homework exercise checking that $\mathrm{Mat}_n(\mathbb{K}) \otimes \mathrm{Mat}_m(\mathbb{K}) \cong \mathrm{Mat}_{mn}(\mathbb{K})$. $\square$