This document is a **transcript** of the lecture, so is more like an abbreviated set of lecture slides than complete lecture notes. For the latter, **consult the textbook** listed on the course webpage.

1 algebras and representations

1.1 Associative Algebras

Setup: Let K be a field. Assume that K is algebraically closed unless noted otherwise.

Note that if K is algebraically closed, then every linear map $K^n \to K^n$ has an eigenvalue in K.

Usually we take $K = \mathbb{C}$, the complex numbers, or $K = \overline{\mathbb{F}_q}$, the algebraic closure of the finite field $\mathbb{F}_q$.

Definition 1.1 (associative algebra)**.** An associative algebra (over K) is a K-vector space A with a bilinear map $A \times A \to A$, written $(a, b) \mapsto a \cdot b$ or ab, that is associative in the sense that $a(bc) = (ab)c$ for all $a, b, c \in A$.

Here, **bilinear** means that the following properties hold for all $a, b, c \in A$ and $\lambda \in K$:

- $(a + b)c = ac + bc$
- $a(b + c) = ab + ac$
- $(\lambda a)b = a(\lambda b) = \lambda(ab)$

Because the product is associative, any way of parenthesizing an iterated product $a_1 a_2 a_3 ... a_n$ with $a_i \in A$ gives the same result, so we can just omit the parentheses in such expressions.

Definition 1.2 (unit)**.** A unit for an associative algebra A is an element $1 \in A$ with $1a = a1 = a$ for all $a \in A$.

Proposition 1.3. If A has a unit then it is unique.

Proof. If 1 and 1' are units for A, then $1 = 11' = 1'$ since $a = a1'$ and $1a = a$. □

From now on, an **algebra** (over K) means a nonzero, associative algebra that has a unit. A subalgebra of an algebra is a subspace containing the unit that is closed under multiplication.

Example 1.4.

Let n be a positive integer. Here are some algebras:

- (Trivial algebra) The field K is itself an algebra. This is the smallest possible algebra, up to isomorphism, since the zero vector space is not an algebra.
- (Polynomial algebra) The set $K[x_1, x_2, ..., x_n]$ of polynomials in commuting variables x_i with coefficients in K is an algebra with unit 1. This algebra is commutative, meaning $fg = gf$ for all elements f and g.
- (Endomorphism algebra) Let V be a K-vector space. Let End V be the vector space of K-linear maps $V \to V$. This is an algebra for the product given by composition $\rho_1\rho_2 \overset{\text{def}}{=} \rho_1 \circ \rho_2$ for $\rho_i : V \to V$. The unit is the identity map $id_V : V \to V$.
  Aside: the vector space of all maps $V \to V$ is also an algebra with the same product and unit, but this is an unreasonably high-dimensional object that is not of much interest.
- (Free algebra) The set $K\langle X_1, X_2, ..., X_n\rangle$ of polynomials in non-commuting variables $X_1, X_2, ..., X_n$ is also an algebra.

- (Group algebra) Given a group G. Let $K[G]$ be the K-vector space with basis $\{a_g : g \in G\}$. This becomes an algebra for the bilinear multiplication that has $a_g a_h = a_{gh}$ for $g, h \in G$. Unit is a_{1_G} where 1_G is the unit for G.

Definition 1.5 (morphism)**.** A morphism $f : A \to B$ of algebras (over K) is a K-linear map such that

- $f(ab) = f(a)f(b)$ for $a, b \in A$
- $f(1_A) = 1_B$

We say f is an isomorphism if there exists a morphism $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$. This occurs if and only if f is a bijection.

Example 1.6. There is a unique morphism $K\langle X_1, X_2, ..., X_n\rangle \to K[x_1, x_2, ..., x_n]$ that sends each variable $X_i \mapsto x_i$ (i.e., that lets the variables commute). In fact, if A is any algebra and we choose some elements $a_1, a_2, ..., a_n$, then there is a unique morphism $K\langle X_1, X_2, ..., X_n\rangle \to A$ sending each $X_i \mapsto a_i$.

Example 1.7. The field K viewed as a K-algebra is **initial** in the category of K-algebras: there is a unique morphism $K \to A$ for any K-algebra A.

1.2 Representations

Definition 1.8 (representation)**.** Let A be an algebra over K. A representation of A is a pair (p, V) where V is a K-vector space and

$$p : A \to \mathrm{End}(V)$$

is an algebra morphism.

If V is a vector space (over K), then

$$\mathrm{End}(V) \overset{\text{def}}{=} \{\text{linear maps } V \to V\}$$

is an algebra, where the product is composition

$$\rho_1\rho_2 \overset{\text{def}}{=} \rho_1 \circ \rho_2,$$

and the unit is $\mathrm{id}_V : V \to V$.

Remark 1.9. Sometimes we will say that the map $p : A \to \mathrm{End}(V)$ is a representation. If p is known implicitly, we may also refer to V as a representation of A.

Definition 1.10 (left A-module)**.** A left A-module is a vector space V with a bilinear map

$$A \times V \to V, \quad (a, v) \mapsto a \cdot v,$$

such that:

- $1 \cdot v = v$ for all $v \in V$
- $a \cdot (b \cdot v) = (ab) \cdot v$ for all $a, b \in A$ and $v \in V$

Theorem 1.11. Representations of A are the same as left A-modules in the following sense:

- If (p, V) is a representation, then setting

$$a \cdot v \overset{\text{def}}{=} p(a)(v)$$

for $a \in A$ and $v \in V$ makes V into a left A-module.

- If V is a left A-module, then setting
$$p(a)(v) \stackrel{\text{def}}{=} a \cdot v$$
for $a \in A$ and $v \in V$ defines a representation $p : A \to \operatorname{End}(V)$.
Moreover, these operations are inverses of each other.

Definition 1.12 (opposite algebra)**.** Let A^{op} be the same vector space as A but with multiplication defined by
$$a *_{\mathrm{op}} b = ba \quad \text{for } a, b \in A.$$
This gives another algebra with the same unit as A, known as the opposite algebra.

It is instructive to check the associativity of $*_{\mathrm{op}}$ directly:
$$a *_{\mathrm{op}} (b *_{\mathrm{op}} c) = a *_{\mathrm{op}} (cb) = (cb)a = c(ba) = (a *_{\mathrm{op}} b) *_{\mathrm{op}} c.$$

Definition 1.13 (right A-module)**.** A right A-module is a vector space V with a bilinear map
$$V \times A \to V, \quad (v, a) \mapsto v \cdot a,$$
such that:

- $v \cdot 1 = v$ for all $v \in V$,
- $(v \cdot a) \cdot b = v \cdot (ab)$ for all $a, b \in A$ and $v \in V$.

Representations of A^{op} are the same as right A-modules, in the same sense as above. If A is commutative, then $A = A^{\mathrm{op}}$. In this case, left A-modules are the same as right A-modules.

Example 1.14. Here are two common representations:

- (**Zero representation**) If $V = 0$, then $\operatorname{End} V$ consists of the unique map $0 \to 0$, and the unique map $A \to \operatorname{End} V = \{0 \to 0\}$ is an algebra morphism.
- (**Regular representation**) Define $\rho : A \to \operatorname{End} V$ by $\rho(a)(b) = ab$ for $a, b \in A$; then (ρ, A) is a representation of A.

If $A = K$, then any K-vector space is a left A-module and so affords a representation.

Definition 1.15 (subrepresentation)**.** Suppose (ρ, V) is a representation of A. A subrepresentation of (ρ, V) is a subspace $W \subset V$ such that $\rho(a)(W) \subseteq W$ for all $a \in A$. Note that 0 and V itself are always subrepresentations. We say that (ρ, V) is **irreducible** if $V \neq 0$ and there are no other subrepresentations.

Definition 1.16. If V is a left A-module, then a submodule is a subspace $W \subset V$ such that $aw \in W$ for all $a \in A$ and $w \in W$. Under the correspondence between representations and left modules described above, subrepresentations correspond to submodules. In this sense, subrepresentations are the same thing as submodules.

1.3 Morphisms of representations

Definition 1.17 (morphism of representations)**.** Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of A. A morphism
$$\phi : (\rho_1, V_1) \to (\rho_2, V_2)$$
is a linear map $\phi : V_1 \to V_2$ such that
$$\phi\big(\rho_1(a)(v)\big) = \rho_2(a)\big(\phi(v)\big)$$

for all $a \in A$ and $v \in V_1$. This property holds precisely when the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\phi"'] & V_2 \end{tikzcd}$$

commutes for all $a \in A$.

We say that ϕ is an isomorphism if ϕ is a bijection.

Proposition 1.18 (Schur's Lemma)**.** For this result, K may be any field, not necessarily algebraically closed. Let (ρ_1, V_1) and (ρ_2, V_2) be representations of A. Suppose

$$\phi : (\rho_1, V_1) \to (\rho_2, V_2)$$

is a nonzero morphism.

- If (ρ_1, V_1) is irreducible then ϕ is injective.
- If (ρ_2, V_2) is irreducible then ϕ is surjective.
- If both representations are irreducible then ϕ is an isomorphism.

Proof. Check that

$$\ker \phi = \{v \in V_1 : \phi(v) = 0\} \subset V_1 \quad \text{and} \quad \operatorname{Im} \phi = \{\phi(v) : v \in V_1\} \subset V_2$$

are subrepresentations. But $\ker \phi \neq V_1$ and $\operatorname{Im} \phi \neq 0$ if ϕ is nonzero. The result therefore follows since only 0 and V can be subrepresentations of an irreducible representation (p, V). □

For the last two results we go back to assuming that K is algebraically closed.

Corollary 1.19. Assume K is algebraically closed and (ρ, V) is an irreducible representation of A with V finite dimensional. Suppose $\phi : (\rho, V) \to (\rho, V)$ is a morphism. Then there exists a scalar $\lambda \in K$ such that

$$\phi(v) = \lambda v \quad \text{for all } v \in V,$$

that is, $\phi = \lambda \cdot \mathrm{id}_V$ is a scalar map.

Proof. As K is algebraically closed, there must be an eigenvalue for ϕ, i.e. there must be some $\lambda \in K$ such that $\phi - \lambda \cdot \mathrm{id}_V$ is not invertible. But $\phi - \lambda \cdot \mathrm{id}_V$ is another morphism $(\rho, V) \to (\rho, V)$. So by Schur's Lemma we must have

$$\phi - \lambda \cdot \mathrm{id}_V = 0.$$

□

Corollary 1.20. If K is algebraically closed and A is commutative, then every irreducible representation (ρ, V) of A has $\dim V = 1$.

Proof. If (ρ, V) is a representation, then the map $\rho(a) : V \to V$ for any $a \in A$ is a morphism $(\rho, V) \to (\rho, V)$ since A is commutative. By Corollary 1.19, we must have

$$\rho(a) = \lambda \cdot \mathrm{id}_V$$

for some $\lambda \in K$. But this applies to every $a \in A$, so every subspace of V is a subrepresentation. Therefore V is irreducible if and only if $\dim V = 1$. □

Exercise 1.21. (Important counterexamples).
Assume $K = \mathbb{R}$ is the field of real numbers, which is not algebraically closed. Then 1.19 and 1.20 can both fail, in the following way:

Let

$$A \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a,\, b \in \mathbb{R} \right\} \stackrel{\text{def}}{=} V$$

As $\mathbb{R}$-algebras, $A \simeq \mathbb{C}$. Let

$$\rho : A \to \mathrm{End}(V) = \mathrm{End}(A)$$

be the regular representation, i.e.,

$$\rho(y)(z) \stackrel{\text{def}}{=} yz.$$

Every $0 \neq z \in A$ is invertible, so (ρ, V) is irreducible with (real) dimension 2. This contradicts 1.20 above since A is commutative.

Define

$$\phi \left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) = \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix}$$

which is multiplication by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This is a morphism

$$\phi : (\rho, V) \to (\rho, V)$$

since A is commutative, but it is not a scalar map for the scalars $K = \mathbb{R}$, contradicting 1.19.

2 ideals, quotients, generators, relations

2.1 Indecomposable representations

Let A be an algebra over K, not necessarily algebraically closed.

Definition 2.1 (direct sum representation)**.** Suppose (ρ_1, V_1) and (ρ_2, V_2) are representations of A. Then we can form the direct sum representation

$$(\rho_1, V_1) \oplus (\rho_2, V_2) \stackrel{\text{def}}{=} (\rho_1 \oplus \rho_2, V_1 \oplus V_2),$$

where $(\rho_1 \oplus \rho_2)(a)(v_1 + v_2) \stackrel{\text{def}}{=} \rho_1(a)(v_1) + \rho_2(a)(v_2)$ for $a \in A$, $v_1 \in V_1$ and $v_2 \in V_2$, and $V_1 \oplus V_2$ is a direct sum of vector spaces.

Note that $(\rho_1, V_1) \oplus (\rho_2, V_2) \cong (\rho_2, V_2) \oplus (\rho_1, V_1)$.

Definition 2.2 (indecomposable)**.** A representation (ρ, V) is indecomposable if it is not isomorphic to $(\rho_1, V_1) \oplus (\rho_2, V_2)$ for any nonzero representations (ρ_i, V_i). This occurs if and only if (ρ, V) does not have two nonzero subrepresentations $W_1, W_2 \subset V$ with $V = W_1 \oplus W_2$ as any internal direct sum.

Remark 2.3. If $W_1, W_2 \subset V$ are subspaces then writing

(a) $V = W_1 \oplus W_2$

is just an abbreviation for

(b) it holds that $V = W_1 + W_2$ and $0 = W_1 \cap W_2$.

In general, the direct sum $W_1 \oplus W_2$ is some new vector space satisfying a universal property, with canonical inclusions of W_1 and W_2. When (b) holds, the ambient vector space V satisfies these conditions so can be identified with $W_1 \oplus W_2$.

Note that irreducible $\implies$ indecomposable, but not vice versa.

Example 2.4. Consider the (commutative) polynomial algebra $A = K[x]$. What are the irreducible representations of A?

Choose a linear map $L : V \to V$, where V is a vector space.

Define $\rho_L : K[x] \to \mathrm{End}(V)$ by formula such that

$$\rho_L(f(x)) = f(L) \iff \rho_L(a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0) = a_n L^n + \ldots + a_2 L^2 + a_1 L + a_0 I.$$

Then (ρ_L, V) is a representation of $K[x]$.

Every representation of $K[x]$ must arise via this construction because every algebra morphism $A \to B$ is uniquely determined by the image of the variable x. It is possible that different choices of L might give isomorphic representations (ρ_L, V), however.

The representation (ρ_L, V) is irreducible if and only if $\dim V = 1$ since K is algebraically closed and $K[x]$ is commutative.

What are the indecomposable representations of $K[x]$?

Choose $\lambda \in K$ and an integer $n \geq 1$. Define $J_{\lambda,n} : K^n \to K^n$ to be the linear map with matrix

$$\begin{bmatrix} \lambda & 1 & & 0 & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{bmatrix}$$

For example, $J_{\lambda,3} =$ linear map with matrix $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

Then $(\rho_{J_{\lambda,n}}, K^n)$ is indecomposable (but not irreducible if $n > 1$), and every indecomposable representation of $K[x]$ is isomorphic to one of these representation, by the uniqueness of Jordan canonical form.

Moreover, it holds that $(\rho_{J_{\lambda,n}}, K^n) \cong (\rho_{J_{\lambda',n'}}, K^{n'})$ if and only if $n = n'$ and $\lambda = \lambda'$.

These statements are not self-evident; their proofs requires a lot of linear algebra work.

2.2 Group Representations

Suppose G is a group. Given a vector space V, let $\mathrm{GL}(V)$ be the group of invertible linear maps $V \to V$.

Definition 2.5 (group representation)**.** A group representation of G is a pair (ρ, V) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism.

Group representations are the same as representations of the corresponding group algebra.

Recall that the **group algebra** is $K[G] = K\text{-span}\{a_g : g \in G\}$ where $a_g a_h = a_{gh}$.

We can turn any group representation (ρ, V) for G into a representation of $K[G]$ by setting $\rho(a_g) = \rho(g)$ and extending by linearity.

Conversely, if (ρ, V) representation of $K[G]$ then every $\rho(a_g) \in \mathrm{GL}(V)$ for $g \in G$ is invertible and

$$g \mapsto \rho(a_g) \in \mathrm{GL}(V)$$

is a group homomorphism $G \to \mathrm{GL}(V)$. This holds since for every invertible $a \in A$ in any algebra, we have $\rho(a)\rho(a^{-1}) = \rho(1_A) = \mathrm{id}_V$ for any representation (ρ, V).

2.3 Ideals in algebras

Let A be an algebra.

Definition 2.6 (ideal)**.** A left ideal in A is a subspace $I \subset A$ with $aI \stackrel{\text{def}}{=} \{ai : i \in I\} \subseteq I$ for all $a \in A$, a right ideal in A is a subspace $I \subset A$ with $Ia \subset I$ for all $a \in A$. A two-sided ideal in A is a subspace that is both a left and right ideal. All three notions coincide if A is commutative.

Remark 2.7. Left ideals are the same as subrepresentations of the regular representation of A and right ideals are the same as subrepresentations of the regular representation of A^{op}.

Remark 2.8. The subspaces 0 and A are always two-sided ideals. If these are the only two-sided ideals then A is simple.

Example 2.9. The algebra $\mathrm{Mat}_{n\times n}(K)$ is simple. To check this, we need to show that if $I \subset \mathrm{Mat}_{n\times n}(K)$ is a nonzero two-sided ideal then every $n \times n$ matrix is in I. If there is some elementary matrix $E_{jk} \in I$, then every other elementary matrix is obtained as $E_{il} = E_{ij}E_{jk}E_{kl} \in I$ so any linear combination of elementary matrices is in I, which means that every $n \times n$ matrix is in I. So it is enough to show that I contains some elementary matrix. As I is nonzero, there is some $0 \neq M \in I$ with some nonzero entry $M_{jk} \neq 0$, and then we have $E_{jk} = \frac{1}{M_{jk}} E_{jj} M E_{kk} \in I$ as needed.

Example 2.10. If $\phi : A \to B$ is an algebra morphism then the *kernel*

$$\ker \phi \stackrel{\text{def}}{=} \{a \in A : \phi(a) = 0\}$$

is a two-sided ideal. The kernel is always a subspace, and if $\phi(a) = 0$ then $\phi(xay) = \phi(x)\phi(a)\phi(y) = 0$ for all $x, y \in A$. Taking $x = 1$ shows that A is right ideal and taking $y = 1$ shows that A is a left ideal, so it is a two-sided ideal.

Example 2.11. If $S \subset A$ is any set, then we define $\langle S \rangle$ to be the intersection of all two-sided ideals in A containing S. We call this the **two-sided ideal generated by** S. Exercise: show that all elements of $\langle S \rangle$ have the form $a_1 s_1 b_1 + \ldots + a_n s_n b_n$ for some $n \geq 0$ and some $a_i, b_i \in A$, $s_i \in S$.

Example 2.12. A maximal left/right/two-sided ideal $I \subsetneq A$ is an ideal properly contained in exactly one other left/right/two-sided ideal (namely A itself). One can use Zorn's lemma to show that every ideal is contained in a maximal ideal. (Zorn's lemma is only needed if A is infinite-dimensional.)

Definition 2.13 (quotient algebra)**.** Assume I is two-sided ideal in an algebra A with $I \neq A$. Then the quotient vector space

$$A/I = \{a + I : a \in A\}$$

where $a + I \stackrel{\text{def}}{=} \{a + i : i \in I\}$ is an algebra for the multiplication defined by

$$(a + I)(b + I) = ab + I \quad \text{for } a, b \in A.$$

The unit is $1 + I$. There is something to check to make sure that the above multiplication is well-defined. This is a standard exercise. The linear map $\pi : A \to A/I$ with $\pi(a) = a + I$ is an algebra morphism.

Definition 2.14 (quotient representation)**.** If (ρ_V, V) is a representation of A and $W \subset V$ is a subrepresentations, then we define the $\rho_{V/W} : A \to \mathrm{End}(V/W)$ by the formula

$$\rho_{V/W}(a)(x + W) = \rho_V(a)(x) + W \text{ for } a \in A, x \in V.$$

Then $(\rho_{V/W}, V/W)$ is a representation of A, called the quotient representation.

If $I \subset A$ is a left ideal, then A/I is a representation of A via this construction.

Equivalently, A/I is a left A-module for the action $a \cdot (b + I) \stackrel{\text{def}}{=} ab + I$ for $a, b \in A$.

2.4 Generators and relations

Recall that $K\langle X_1, \ldots, X_n\rangle$ is the **free algebra** of polynomials in noncommuting variables.

If $f_1, \ldots, f_m \in K\langle X_1, \ldots, X_n\rangle$ then we can consider the quotient algebra

$$K\langle X_1, X_2, \ldots, X_n\rangle / \langle\{f_1, f_2, \ldots, f_m\}\rangle,$$

which we often denote by writing

$$K\langle X_1, X_2, \ldots X_n \mid f_1 = f_2 = \ldots f_m = 0\rangle.$$

We think of the elements of this quotient are polynomials as usual, but we can replace expressions equal to f_i by zero.

Remark 2.15. Technically, if $I = \langle\{f_1, f_2, \ldots, f_m\}\rangle$ then elements of $K\langle X_1, X_2, \ldots, X_n\rangle/I$ are cosets of the form $f + I$. Usually we write things by dropping the "$+I$" part, even though this can make it ambiguous whether f belongs to $K\langle X_1, X_2, \ldots, X_n\rangle$ or the quotient.

Example 2.16. The **Weyl algebra** is

$$K\langle x, y \mid yx - xy - 1 = 0\rangle = K\langle x, y \mid yx - xy = 1\rangle.$$

In the Weyl algebra, we have $yx = xy + 1$ and $xyx = x(xy + 1) = x^2y + x = (yx - 1)x = yx^2 - x$.

Example 2.17. The **q-Weyl algebra** for a fixed nonzero element $q \in K$ is

$$K\langle x, x^{-1}, y, y^{-1} \mid yx = qxy \text{ and } xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\rangle.$$

The second set of relations ensures that x, x^{-1} and y, y^{-1} are inverses of each other.

3 quivers and Lie algebras

3.1 Weyl algebra

Things work out nicely for the Weyl algebra $A = K\langle x, y \mid yx = xy + 1\rangle$.

Proposition 3.1. A basis for the Weyl algebra is $\{x^i y^j : i, j \geq 0\}$.

Proof. It is easy to see that the set spans algebra, since

$$x^{i_1} y^{j_1} x^{i_2} y^{j_2} \ldots x^{i_k} y^{j_k} = x^{i_1+i_2+\ldots+i_k} y^{j_1+j_2+\ldots+j_k} + \text{(lower degree terms)}$$

via repeated substitutions $yx = xy + 1$.

To show linear independence, assume $\mathrm{char}(K) = 0$.

(The argument when $\mathrm{char}(K) > 0$ is similar but not as elegant; see the textbook for details.)

Consider the polynomial ring $K[z]$. For $f \in K[z]$, define $x \cdot f = zf$ and $y \cdot f = \frac{df}{dz}$.

There is a unique left A-module structure on $K[z]$ with these formulas, because

$$y \cdot (x \cdot f) = y \cdot (zf) = \frac{d}{dz}(zf) = f + z\frac{df}{dz} = f + x \cdot (y \cdot f),$$

which is equivalent to $yx = xy + 1$.

Now suppose $c_{ij} \in K$ are such that $\sum_{i,j} c_{ij} x^i y^j = 0$ in A.

Let $L = \sum_{i,j} c_{ij} z^i \left(\frac{d}{dz}\right)^j$ be a differential operator on $K[z]$. Then

$$L(f) = \left(\sum_{i,j} c_{ij} x^i y^j\right) \cdot f = 0 \text{ for all } f \in K[z].$$

But we can write $L = \sum_{j=0}^{r} Q_j(z) \left(\frac{d}{dz}\right)^j$ for some polynomials $Q_j(z) \in K[z]$.

Now observe that

$$\begin{aligned} L(1) &= Q_0(z) = 0 \\ L(z) &= Q_0(z)z + Q_1(z) = Q_1(z) = 0 \\ L(z^2) &= Q_0(z)z^2 + Q_1(z)z + Q_2(z) = Q_2(z) = 0 \\ &\vdots \end{aligned}$$

Thus we have $Q_0 = Q_1 = \ldots = Q_r = 0 \Longrightarrow c_{ij} = 0$ for every i, j, which proves that elements $x^i y^j$ must be linearly independent. □

Example 3.2. The **q-Weyl algebra** is

$$A = K\langle x, x^{-1}, y, y^{-1} \mid yx = qxy, xx^{-1} = x^{-1}x = 1, yy^{-1}y^{-1}y = 1\rangle.$$

Here, $q \in K$ is a fixed nonzero element.

We require q to be nonzero, since if $q = 0$ then $x = y = 0$ so $A = 0$:

$$yx = 0 \Longrightarrow y^{-1}yx = x = 0 \text{ and } yxx^{-1} = y = 0.$$

Proposition 3.3. If $q \neq 0$, then a basis for the q-Weyl algebra is $\{x^i y^j : i, j \in \mathbb{Z}\}$

3.2 Quiver representations

Quivers are another source of algebra representations.

Definition 3.4 (quiver)**.** A quiver $Q = (I, E)$ is a directed graph with self-loops and multiple edges allowed. Here, I is the set of vertices in Q and E is the multi-set of directed edges $i \to j$.

A **multi-set** is, informally, a set allowing repeated elements. This can be viewed formally as a map from an arbitrary set to the set of positive integers $\{1, 2, 3, \ldots\}$.

Example 3.5. We draw the quiver $Q = (\{a, b, c, d\}, \{a \to b, c \to b, d \to b\})$ as

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
& d \arrow[u] &
\end{tikzcd}$$

Example 3.6. The quiver $Q = (I, E)$, where $I = \{1, 2, 3, 4, 5\}$ and

$$E = \{1 \to 1, 1 \to 2, 1 \to 2, 2 \to 1, 1 \to 4, 3 \to 5\},$$

can be drawn as

$$\begin{tikzcd}
1 \arrow[loop left] \arrow[r, shift left=2] \arrow[r] \arrow[dr] & 2 \arrow[l, shift left=2] \\
3 \arrow[dr] & 4 \\
& 5
\end{tikzcd}$$

Definition 3.7 (representation of a quiver)**.** A representation (V_*, ρ_*) of a quiver $Q = (I, E)$ is an assignment of a vector space V_i for each $i \in I$ and a linear map $\rho_{ij} : V_i \to V_j$ for each edge $i \to j$ in E.

Why is this relevant? Quiver representations are natural to consider because they contain the same data as an arbitrary diagram of linear maps between vector spaces. Moreover, there is a related **path algebra** whose algebra representations are in bijection with quiver representations.

Definition 3.8 (path algebra of a quiver)**.** The path algebra Path_Q of a quiver $Q = (I, E)$ is the K-vector space with a basis given by all directed paths in Q, including trivial paths p_i for each $i \in I$, with multiplication of paths given by

$$(i_0 \to i_1 \to \ldots \to i_n) \cdot (j_0 \to j_1 \to \ldots \to j_m) \stackrel{\text{def}}{=} \begin{cases} j_0 \to j_1 \to \ldots \to j_m \to i_1 \to i_2 \ldots i_n & \text{if } j_m = i_0 \\ 0 & \text{if } j_m \neq i_0 \end{cases}$$

If Q has a finite set of vertices, then the element $\sum_{i \in I} p_i$ is the unit in Path_Q.

Proposition 3.9. We can translate between representations of Q and Path_Q:

- Suppose (ρ, V) is a representation of Path_Q.

  Define a representation of Q by setting $V_i = \rho(p_i)(V)$ for $i \in I$ and

  $$\rho_{ij} = \rho(i \to j)|_{V_i} : V_i \to V_j$$

  for edges $i \to j$ in E.

  The definition of ρ_{ij} makes sense as a map $V_i \to V_j$ since

  $$\begin{aligned}\rho(i \to j)(V_i) &= \rho(i \to j) \circ \rho(p_i)(V) = \rho(i \to j \cdot p_i)(V) = \rho(i \to j)(V) \\ &= \rho(p_j \cdot i \to j)(V) = \rho(p_j) \circ \rho(i \to j)(V) \subset \rho(p_j)(V) = V_j.\end{aligned}$$

- Suppose (V_*, ρ_*) is a quiver representation of Q. Form a representation of Path_Q by setting

  $$V = \bigoplus_{i \in I} V_i$$

  and let $\rho(i_0 \to i_1 \to \dots i_m)$ be the unique linear map $V \to V$ that sends

  $$\begin{cases} V_j = V_{i_0} \xrightarrow{\rho_{i_0 i_1}} V_{i_1} \xrightarrow{\rho_{i_1 i_2}} \dots \xrightarrow{\rho_{i_{m-1} i_m}} V_{i_m} & \text{if } i_0 = j \\ V_j \to 0 & \text{if } i_0 \neq j. \end{cases}$$

  In particular, $\rho(p_i)$ is the projection $V \to V_i$. Then (ρ, V) is a representation of Path_Q.

The above operationsare inverses of each other.

3.3 Lie algebra representations

Lie algebras are another sources of "representations" that can be viewed as a special case of algebra representations. Despite the name, Lie algebras are not "algebras" according to our definition, since their products are not associative. Here is the actual definition:

Let $\mathfrak{g}$ be a vector space over K.

Assume $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map satisfying $[a, a] = 0$ for all $a \in \mathfrak{g}$.

This property implies that $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$, since

$$0 = [a + b, a + b] = [a, a + b] + [b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = [a, b] + [b, a].$$

Definition 3.10 (Lie algebra)**.** We say that $\mathfrak{g}$ is a Lie algebra relative to the bracket $[\cdot, \cdot]$ if the **Jacobi identity** holds:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \text{ for all } a, b, c \in \mathfrak{g}.$$

A Lie subalgebra of a Lie algebra is a subspace closed under the bracket.

Example 3.11. If A is any associative algebra, then $[a, b] \overset{\text{def}}{=} ab - ba$ makes A into a Lie algebra.

Example 3.12. Let $\mathrm{Der}(A)$ be a vector space of linear maps $D : A \to A$ (for an algebra A) satisfying $D(ab) = aD(b) + D(a)b$. This is a Lie algebra for the bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Example 3.13. If V is any vector space then we write $\mathfrak{gl}(V)$ for the *general linear Lie algebra* obtained by giving the vector space $\mathrm{End}(V)$ the bracket $[f, g] = f \circ g - g \circ f$. This is a special case of Example 4.2.

Theorem 3.14 (Ado's Theorem)**.** If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.

Definition 3.15 (morphism of Lie algebras)**.** A morphism of Lie algebras is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\phi([a, b]) = [\phi(a), \phi(b)] \text{ for all } a, b \in \mathfrak{g}.$$

Definition 3.16 (representation of a Lie algebra)**.** A representation of a Lie algebra $\mathfrak{g}$ is a pair (ρ, V) where V is a vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra morphism.

Example 3.17. The *adjoint representation* of $\mathfrak{g}$ is $(\rho, \mathfrak{g})$ where $\rho(a)(b) = [a, b]$ for $a, b \in \mathfrak{g}$.

Definition 3.18 (universal enveloping algebra)**.** For any Lie algebra $\mathfrak{g}$, there is a related algebra, called the **universal enveloping algebra** $\mathfrak{U}(\mathfrak{g})$, such that there is a bijective correspondence between the Lie algebra representations of $\mathfrak{g}$ and the algebra representations of $\mathfrak{U}(\mathfrak{g})$.

If $\mathfrak{g}$ has a basis $\{x_i\}_{i\in I}$ and $c_{ij}^k \in K$ are the coefficients such that $[x_i, x_j] = \sum_k c_{ij}^k x_k$ for each $i, j \in I$, then $\mathfrak{U}(\mathfrak{g})$ may be defined via generators and relations as the quotient algebra

$$\mathfrak{U}(\mathfrak{g}) = K \left\langle x_i \text{ for } i \in I \mid x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k \text{ for all } i, j \in I \right\rangle.$$

4 tensor products

4.1 Tensor products of vector spaces

Let V and W be two K-vector spaces. Their direct product is simply the set of pairs

$$V \times W = \{(v, w) : v \in w, b \in W\}.$$

Definition 4.1 (tensor product)**.** This object is just a set, not a vector space. Define the free product $V * W$ to be the K-vector space with $V \times W$ as a basis. Each element of $V * W$ is a finite linear combination of pairs $(v, w) \in V \times W$.

One way to define the *tensor product* of V and W is as the quotient vector space

$$V \otimes W \stackrel{\text{def}}{=} (V * W)/\mathcal{I}_{V,W}$$

where $\mathcal{I}_{V,W}$ is the subspace spanned by all elements of the form

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$,
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$,
- $(av, w) - a(v, w)$, or
- $(v, aw) - a(v, w)$,

for any $a \in K$, $v_1, v_2, v \in V$, and $w_1, w_2, w \in W$.

If $x \in V$ and $y \in W$, then we write $x \otimes y \in V \otimes W$ for the image of the pair $(x, y) \in V \times W \subset V * W$ under the quotient map $V * W \to V \otimes W$. This means that

$$x \otimes y \stackrel{\text{def}}{=} (x, y) + \mathcal{I}_{V,W}$$

if we view elements of a vector space quotient a cosets of a subspace.

We refer to $x \otimes y$ as a **pure tensor**. Not all elements of $V \otimes W$ are pure tensors, but every element is a finite linear combination of pure tensors.

We can manipulate pure tensors without changing their value in $V \otimes W$ using the following identities:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad (cv) \otimes w = c(v \otimes w) = v \otimes (cw)$$

for $v_1, v_2, v \in V$, $w_1, w_2, w \in W$, and $c \in K$.

These equations hold because the differences between the two sides belong to the subspace $\mathcal{I}_{V,W}$.

This means that we can have $x \otimes y = x' \otimes y'$ when $x \neq x'$ and $y \neq y'$.

A simple example is when $x' = -x \in V$ and $y' = -y \in W$.

Exercise 4.2. (Important to do once).
If $\{v_i : i \in I\}$ is a basis of V and $\{w_j : j \in J\}$ is a basis of W then the set of pure tensors $\{v_i \otimes w_j : (i, j) \in I \times J\}$ is a basis of $V \otimes W$.

Example 4.3. If U, V, and W are K-vector spaces, then there is a unique isomorphism

$$(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

that sends $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$ for each $u \in U$, $v \in V$, and $w \in W$.

As a result of this exercise, there is a canonical isomorphism between any way of forming the tensor product between a finite sequence of vector spaces (in principle, each way requires us to choose a parenthesization of the factors, since we can only tensor two spaces at a time). For example:

$$V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \cong V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) \cong (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \cong ((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \cong (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4.$$

In view of this, we will ignore the issue of parenthesization and just define

$$V^{\otimes 0} \stackrel{\text{def}}{=} K \quad \text{and} \quad V^{\otimes n} \stackrel{\text{def}}{=} V \otimes \cdots \otimes V \ (n \text{ factors}).$$

4.2 Tensor products of linear maps

Definition 4.4 (tensor products of linear maps)**.** If $f \in \mathrm{Hom}(V, V')$ and $g \in \mathrm{Hom}(W, W')$ are two linear maps then their tensor product is the unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ that acts on pure tensors as

$$v \otimes w \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

There are some things to check to make sure that this is well-defined. Since $V \times W$ is a basis for $V * W$, there is certainly a unique linear map $f * g : V * W \to V' \otimes W'$ that sends

$$(v, w) \mapsto f(v) \otimes g(w) \quad \text{for all } v \in V \text{ and } w \in W.$$

We want know that the map $f * g$ descends to a well-defined map of quotient spaces $V \otimes W \to V' \otimes W'$, since this will give exactly our desired map $f \otimes g$. So we need to verify that $(f * g)(\mathcal{I}_{V,W}) = 0 \subseteq V' \otimes W'$.

To check this, it is enough to show that $f * g$ sends each element in the spanning set for $\mathcal{I}_{V,W}$ to zero. This is some fairly routine algebra. For instance, if $v_1, v_2 \in V$ and $w \in W$ then we have

$$\begin{aligned}(f*g)((v_1+v_2,w)-(v_1,w)-(v_2,w)) &= (f*g)((v_1+v_2,w)) - (f*g)((v_1,w)) - (f*g)((v_2,w)) \\ &= f(v_1+v_2)\otimes g(w) - f(v_1)\otimes g(w) - f(v_2)\otimes g(w) \\ &= f(v_1)\otimes g(w) + f(v_2)\otimes g(w) - f(v_1)\otimes g(w) - f(v_2)\otimes g(w) \\ &= 0\end{aligned}$$

as needed. The calculations showing that $f * g$ kills off the other elements spanning $\mathcal{I}_{V,W}$ are similar.

4.3 Tensor algebra

This is given as a vector space by the infinite direct sum

$$\mathcal{T}V \stackrel{\text{def}}{=} \bigoplus_{n\geq 0} V^{\otimes n}.$$

Remember that the elements of an infinite direct sum are finite sums of elements from the summands.

We view $\mathcal{T}V$ as a K-algebra by defining

$$ab \stackrel{\text{def}}{=} a\otimes b \quad \text{for } a \in V^{\otimes m} \text{ and } b \in V^{\otimes n},$$

and extending by bilinearity. Here we view $a\otimes b \in V^{\otimes(m+n)}$. This product is associative, since the tensor product is associative. The unit of the resulting **tensor algebra** $\mathcal{T}V$ is the field unit $1 = 1_K \in K = V^{\otimes 0}$.

Notice that $\mathcal{T}V$ is an algebra even when $V = 0$, since then $\mathcal{T}V = \mathcal{T}0 = K$.

Exercise 4.5. We may identify tensor algebras with free algebras. Suppose V is finite-dimensional with basis $\{v_1, \ldots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{T}V \xrightarrow{\sim} K\langle X_1, \ldots, X_N\rangle$$

that sends $v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k} \mapsto X_{i_1}X_{i_2}\cdots X_{i_k}$. A similar isomorphisms exists when V is infinite-dimensional, if we allow infinitely-many variables in the free algebra.

We mention three interesting quotients of the tensor algebra: symmetric algebra, exterior algebra, universal enveloping algebras.

4.4 Symmetric algebras

Definition 4.6 (symmetric algebra of V)**.** This is defined by

$$\mathcal{S}V \stackrel{\text{def}}{=} \mathcal{T}V/\langle v\otimes w - w\otimes v : v, w \in V\rangle.$$

Recall that "$\langle v\otimes w - w\otimes v : v, w \in V\rangle$" means the intersection of all two-sided ideals in $\mathcal{T}V$ containing all of the differences $v\otimes w - w\otimes v$ for each $v, w \in V$.

The symmetric algebra $\mathcal{S}V$ is always commutative. We have $\mathcal{T}V \cong \mathcal{S}V$ if and only if $\dim V \leq 1$.

Example 4.7. We may identify symmetric algebras with polynomial algebras. Suppose V is finite-dimensional with basis $\{v_1, \ldots, v_N\}$. Then there is a unique algebra isomorphism

$$\mathcal{S}V \xrightarrow{\sim} K[x_1, \ldots, x_N]$$

that sends $v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k} \mapsto x_{i_1}x_{i_2}\cdots x_{i_k}$. A similar isomorphisms exists when V is infinite-dimensional, if we allow infinitely-many variables in the polynomial algebra.

4.5 Exterior algebras

Definition 4.8 (exterior algebra of V)**.** This is defined by

$$\bigwedge V \stackrel{\text{def}}{=} \mathcal{T}V/\langle v \otimes v : v \in V\rangle.$$

Define $x \wedge y$ to be the image $x \otimes y \in V \otimes V$ under the quotient map $\mathcal{T}V \to \bigwedge V$. Then

$$\begin{aligned} 0 &= (x+y) \wedge (x+y) \\ &= x \wedge x + x \wedge y + y \wedge x + y \wedge y \\ &= x \wedge y + y \wedge x \end{aligned}$$

so $x \wedge y = -y \wedge x$. This shows that the operation $\wedge$ defines an **anti-commutative** product for $\bigwedge V$.

Example 4.9. Choosing a basis for V determines an isomorphism from $\bigwedge V$ to a "polynomial algebra" in which the variables anti-commute in the sense that $x_i x_j = -x_j x_i$.

4.6 Universal enveloping algebras

Definition 4.10 (universal enveloping algebra)**.** If $\mathfrak{g}$ is a Lie algebra then its universal enveloping algebra is the quotient of the tensor algebra

$$U(\mathfrak{g}) \stackrel{\text{def}}{=} \mathcal{T}_{\mathfrak{g}}/\langle x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g}\rangle.$$

The advantage of this formulation is that it does not depend on a choice of basis for V. Our previous definition relied on such a choice, and it was not clear that we got the same algebra for different choices of basis.

4.7 Tensor product of modules

Building on our definition of vector space tensor products, we can now define more general tensor products of modules over a (not necessarily commutative) algebra.

Consider the following setting:

- A, B, C are algebras over the same field K.
- V is a right B-module.
- W is a left B-module.

Then we define $V \otimes_B W$ to be the vector space quotient

$$V \otimes_B W \stackrel{\text{def}}{=} (V \otimes W)/K\text{-span}\{vb \otimes w - v \otimes bw : v \in V, w \in W, b \in B\}.$$

In general, this object only has the structure of a K-vector space.

Specifically, if B is non-commutative, then $V \otimes_B W$ is not naturally a left or right module for B.

We refer to $V \otimes_B W$ as the **tensor product of V and W over B**. If $v \in V$ and $w \in W$ then we write

$$v \otimes_B w \in V \otimes_B W$$

for the image of $v \otimes w \in V \otimes W$ under the quotient map $V \otimes W \to V \otimes_B W$. Notice that if $b \in B$ then

$$vb \otimes_B w = v \otimes_B bw.$$

- V is an (A, B)-bimodule, meaning that
  1. V has both right B-module and left A-module structures;
  2. these structures are compatible in the sense that $(av)b = a(vb)$ for all $a \in A, b \in B, v \in V$.

Assume likewise that

- W is a (B, C)-bimodule, meaning that
  1. W has both left B-module and right C-module structures;
  2. these structures are compatible in the sense that $(bw)c = b(wc)$, for all $b \in B, c \in C, w \in W$.

Then the vector space $V \otimes_B W$ has a (A, C)-bimodule structure defined by

$$\begin{cases} a(v \otimes_B w) \overset{\text{def}}{=} (av) \otimes_B w & \text{if } a \in A \\ (v \otimes_B w)c \overset{\text{def}}{=} v \otimes_B (wc) & \text{if } c \in C \end{cases} \qquad \text{for } v \in V \text{ and } w \in W.$$

The case when $A = B = C$ is worth noting. In this situation, V and W are both (B, B)-bimodules, and the tensor product $V \otimes_B W$ is also a (B, B)-bimodule.

Remark 4.11. If the algebra B is commutative, then left and right B-modules are the same as (B, B)-bimodules (do you see why?), and so we **can** form the tensor product of two left B-modules or two right B-modules. However, this is secretly just doing the (B, B)-bimodule tensor product.

4.8 Diagrammatic de nition of an algebra

Now that we have a good handle on vector space tensor products, we can given an alternate definition of an **algebra**. This consists of a K-vector space A with linear maps $\nabla : A \otimes A \to A$ and $\iota : K \to A$ that make the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\nabla \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \nabla"'] & A \otimes A \arrow[d, "\nabla"] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd} \qquad \begin{tikzcd} K \otimes A \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\nabla"] & A \otimes K \arrow[l, "\mathrm{id} \otimes \iota"'] \arrow[dl, "\simeq"] \\ & A & \end{tikzcd}$$

The diagonal arrows on the right are the linear maps $K \otimes A \to A$ and $A \otimes K \to A$ sending $1_K \otimes a \mapsto a$ and $a \otimes 1_K \mapsto a$ for all $a \in A$. These maps are vector space isomorphisms.

Under this formulation, the product in A is $ab \overset{\text{def}}{=} \nabla(a \otimes b)$ and the unit is $\iota(1_K) \in A$.

One nice feature of this definition is that it naturally suggests the definition of a **coalgebra**: this is the object one gets by repeating the above definition but reversing the direction of all arrows.