

Intermediate complex analysis

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Abstract

This article consists of some notes taken by the author while studying **Intermediate complex analysis**.

References: [1], [2], [3], [4], [5], [6].

References

- [1] Raghavan Narasimhan and Yves Nievergelt. *Complex analysis in one variable*. Springer Science & Business Media, 2012.
- [2] Otto Forster. *Lectures on Riemann surfaces*. Vol. 81. Springer Science & Business Media, 2012.
- [3] John B Conway. *Functions of one complex variable II*. Vol. 159. Springer Science & Business Media, 2012.
- [4] Komaravolu Chandrasekharan. *Elliptic functions*. Vol. 281. Springer Science & Business Media, 2012.
- [5] Steven G Krantz. *Geometric function theory: explorations in complex analysis*. Springer, 2006.
- [6] Lars Valerian Ahlfors and Lars V Ahlfors. *Complex analysis*. Vol. 3. McGraw-Hill New York, 1979.

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1 Preliminaries

1.1 Preliminaries in complex analysis

1.1.1 Cauchy integral formula

Theorem 1.1 (Cauchy Integral Formula 1st form). *Let f be a holomorphic function defined on a neighborhood of $\overline{\Delta(a; r)}$. For any $z \in \Delta(a; r)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta(a; r)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

More generally, if $\Omega \subset \mathbb{C}$ is a domain with piecewise C^1 boundary, then

Corollary 1.1 (Generalized Cauchy Integral Formula).

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Proof. Let $\Omega_\varepsilon = \Omega - \delta(z, \varepsilon)$ where ε is sufficiently small such that $\overline{\delta(z, \varepsilon)} \cap \partial\Omega = \emptyset$. Since $f(\zeta)/(\zeta - z)$ is holomorphic as a function of ζ on Ω_ε . Using Stoke's Theorem and the above theorem yields the result. \square

1.1.2 Residue theorem

Let Ω be defined as above and f be a meromorphic function on $\overline{\Omega}$ without pole on $\partial\Omega$. It automatically implies that f has only finite number of poles. Then, we have

Definition 1.1 (principal part). *Principal part means the sum of all negative-power terms in the Laurent expansion of f at the point a .*

If we take the **Laurent series** of f at point a :

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad (0 < |z-a| < R),$$

then the principal part of f at a is

$$\sum_{n=1}^m \frac{c_{-n}}{(z-a)^n}.$$

(If a is a pole, the negative powers stop at some finite order m ; if a is an essential singularity, there are infinitely many negative-power terms.)

Theorem 1.2 (Residue Theorem).

$$\int_{\partial\Omega} f(z)dz = 2\pi i \sum_j \text{Res}(f; a_j)$$

where a_j 's are poles of meromorphic function f .

Proof. For f meromorphic, let $\{a_j\}$ be the finite set of poles of f in Ω . Consider the function

$$g = f - \sum_j P_j(f)$$

where $P_j(f)$ is the principal part of f at a_j . Then, apply the generalized Cauchy integral formula yields the result. \square

We have a generalization for the residue theorem. Take $\overline{\Omega} \subset \Omega'$, consider a discrete set of points $E \subset \Omega$ such that $E \cap \partial\Omega = \emptyset$. If $f \in \mathcal{O}(\Omega' - E)$, then the Residue theorem still holds for f . The proof follows the same argument as above while the principal parts

$$P_j(f) = \sum_{n=-\infty}^{-1} c_n^{(j)}(z-a_j)^n$$

1.1.3 Winding number

Definition 1.2 (winding number). Let $\Gamma : [a, b] \rightarrow \mathbb{C}$ be a closed piecewise C^1 curve. For $z \notin \text{Im}(\Gamma)$. Define

$$n(\Gamma, z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\zeta}{\zeta - z}$$

$n(\Gamma, z)$ is called the winding number of Γ at z .

Remark 1.1. $n(\Gamma, z)$ is an integer and hence is constant on each connected component of $\mathbb{C} - \text{Im}(\Gamma)$ by continuity.

1.1.4 Cauchy theorem with winding numbers.

Using the winding number, we can generalize the Cauchy integral formula by allowing the domain Ω be multi-connected open subset of \mathbb{C} .

Theorem 1.3 (Homotopy form of Cauchy integral theorem). *Let γ be a closed piecewise C^1 curve. Assume that γ is homotopic to a constant in Ω . Then, for any $z \in \Omega - \text{Im}(\gamma)$,*

$$n(\gamma, z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

We say that a closed continuous curve γ is **homotopic** to Γ if and only if $\exists F : [0, 1] \times [0, 1] \rightarrow \Omega$ such that

$$\begin{cases} F(0, s) = \gamma(s) \\ F(1, s) = \Gamma(s) \end{cases}$$

For our purpose, taking γ, Γ to be piecewise C^1 . We may assume that $\gamma_t(s) = F(t, s)$ is piecewise C^1 in s .

If $\Gamma(s) = p \in \Omega$, then γ is said to be **homotopic to a constant**. In this case, one may assume that $F(t, 0) = p, \forall t \in [0, 1]$ which means that the end point of each curve for each time t is fixed at p .

Proof. Take homotopy $F : [0, 1] \times [0, 1] \rightarrow \Omega$,

$$\begin{cases} F(0, s) = \gamma(s), & \gamma(0) = p = \gamma(1) \\ F(1, s) = p \end{cases}$$

Consider

$$h(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

which is a holomorphic function in ζ defined on $\Omega - \{z\}$. Then, $\text{Res}(h, z) = f(z)$ and the principal part is $f(z)/(\zeta - z)$. Define

$$\begin{aligned} g(\zeta) &= h(\zeta) - \text{principal part at } z \\ &= \frac{f(\zeta)}{\zeta - z} - \frac{f(z)}{\zeta - z} \\ &= \frac{f(\zeta) - f(z)}{\zeta - z} \end{aligned}$$

which has a removable singularity at z . Therefore,

$$\int_{\gamma_t} g(\zeta) d\zeta = \int_{\gamma_1} g(\zeta) d\zeta = 0.$$

□

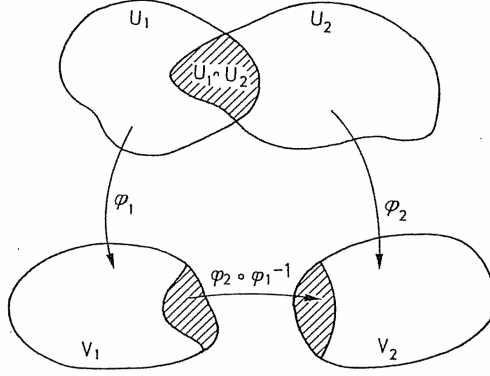


Figure 1: complex manifold

1.2 Preliminaries in Riemann surface

1.2.1 Definition of Riemann surface

In this part we define Riemann surfaces, holomorphic and meromorphic functions on them and also holomorphic maps between Riemann surfaces.

Riemann surfaces are two-dimensional manifolds together with an additional structure which we are about to define. As is well known, an

n -dimensional manifold is a **Hausdorff topological space** X such that every point $a \in X$ has an open neighborhood which is homeomorphic to an open subset of \mathbb{R}^n .

Definition 1.3. Let X be a two-dimensional manifold. A complex chart on X is a homeomorphism $\varphi : U \rightarrow V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i : U_i \rightarrow V_i, i = 1, 2$ are said to be holomorphically compatible if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

A **complex atlas** on X is a system $\mathcal{A} = \{\varphi_i : U_i \rightarrow V_i, i \in I\}$ of charts which are holomorphically compatible and which cover X , i.e., $\bigcup_{i \in I} U_i = X$.

Two complex atlases \mathcal{A} and \mathcal{A}' on X are called **analytically equivalent** if every chart of \mathcal{A} is holomorphically compatible with every chart of \mathcal{A}' .

Remark 1.2. 1. If $\varphi : U \rightarrow V$ is a complex chart, U_1 is open in U and $V_1 := \varphi(U_1)$, then $\varphi|_{U_1} : U_1 \rightarrow V_1$ is a chart which is holomorphically compatible with $\varphi : U \rightarrow V$.

2. Since the composition of biholomorphic mappings is again biholomorphic, one easily sees that the notion of analytic equivalence of complex atlases is an equivalence relation.

Definition 1.4 (complex structure). *By a complex structure on a two-dimensional manifold X we mean an equivalence class of analytically equivalent atlases on X .*

Thus a complex structure on X can be given by the choice of a complex atlas. Every complex structure Σ on X contains a unique maximal atlas \mathcal{A}^* . If \mathcal{A} is an arbitrary atlas in Σ , then \mathcal{A}^* consists of all complex charts on X which are holomorphically compatible with every chart of \mathcal{A} .

Definition 1.5 (Riemann surface). *A Riemann surface is a pair (X, Σ) , where X is a connected two-dimensional manifold and Σ is a complex structure on X .*

One usually writes X instead of (X, Σ) whenever it is clear which complex structure Σ is meant. Sometimes one also writes (X, \mathcal{A}) where \mathcal{A} is a representative of Σ .

If X is a Riemann surface, then by a chart on X we always mean a complex chart belonging to the maximal atlas of the complex structure on X .

Remark 1.3. *Locally a Riemann surface X is nothing but an open set in the complex plane. For, if $\varphi : U \rightarrow V \subset \mathbb{C}$ is a chart on X , then φ maps the open set $U \subset X$ bijectively onto V . However, any given point of X is contained in many different charts and no one of these is distinguished from the others. For this reason we may only carry over to Riemann surfaces those notions from complex analysis in the plane which remain invariant under biholomorphic mappings, i.e., those notions which do not depend on the choice of a particular chart.*

1.2.2 Examples of Riemann Surfaces

Example 1.1 (Complex Plane \mathbb{C}). *Its complex structure is defined by the atlas whose only chart is the identity map $\mathbb{C} \rightarrow \mathbb{C}$.*

Example 1.2 (Domains). *Suppose X is a Riemann surface and $Y \subset X$ is a domain, i.e., a connected open subset. Then Y has a natural complex structure which makes it a Riemann surface. Namely, one takes as its atlas all those complex charts $\varphi : U \rightarrow V$ on X , where $U \subset Y$. In particular, every domain $Y \subset \mathbb{C}$ is a Riemann surface.*

Example 1.3 (Riemann sphere \mathbb{P}^1). *Let $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$, where ∞ is a symbol not contained in \mathbb{C} . Introduce the following topology on \mathbb{P}^1 . The open sets are the usual open sets $U \subset \mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a compact set $K \subset \mathbb{C}$. With this topology \mathbb{P}^1 is a compact Hausdorff topological space, homeomorphic to the 2-sphere S^2 . Set*

$$U_1 := \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}, \quad U_2 := \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

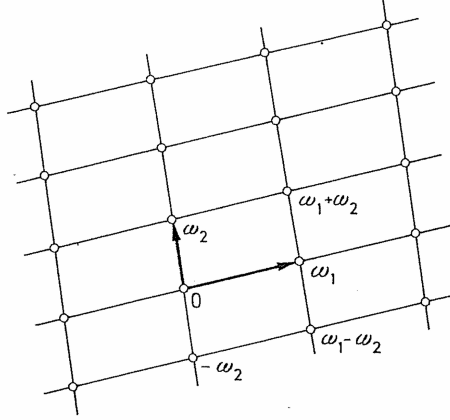


Figure 2: Tori

Define maps $\varphi_i : U_i \rightarrow \mathbb{C}$, $i = 1, 2$, as follows. φ_1 is the identity map and

$$\varphi_2(z) := \begin{cases} 1/z, & \text{for } z \in \mathbb{C}^*, \\ 0, & \text{for } z = \infty. \end{cases}$$

Clearly these maps are homeomorphisms and thus \mathbb{P}^1 is a two-dimensional manifold. Since U_1 and U_2 are connected and have non-empty intersection, \mathbb{P}^1 is also connected.

The complex structure on \mathbb{P}^1 is now defined by the atlas consisting of the charts $\varphi_i : U_i \rightarrow \mathbb{C}$, $i = 1, 2$. We must show that the two charts are holomorphically compatible. But $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C}^*$ and

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad z \mapsto 1/z,$$

is biholomorphic.

Remark 1.4. The notation \mathbb{P}^1 comes from the fact that one may consider \mathbb{P}^1 as the 1-dimensional projective space over the field of complex numbers.

Example 1.4 (Tori). Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over \mathbb{R} . Define

$$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}.$$

Γ is called the **lattice** spanned by ω_1 and ω_2 (Fig. 2). Two complex numbers $z, z' \in \mathbb{C}$ are called *equivalent mod Γ* if $z - z' \in \Gamma$. The set of all equivalence classes is denoted by \mathbb{C}/Γ . Let $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Gamma$ be the canonical projection, i.e., the map which associates to each point $z \in \mathbb{C}$ its equivalence class mod Γ .

Introduce the following topology (the quotient topology) on \mathbb{C}/Γ . A subset $U \subset \mathbb{C}/\Gamma$ is open precisely if $\pi^{-1}(U) \subset \mathbb{C}$ is open. With this topology \mathbb{C}/Γ is a Hausdorff topological

space and the quotient map $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Gamma$ is continuous. Since \mathbb{C} is connected, \mathbb{C}/Γ is also connected. As well \mathbb{C}/Γ is compact, for it is covered by the image under π of the compact parallelogram

$$P := \{\lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in [0, 1]\}.$$

The map π is open, i.e., the image of every open set $V \subset \mathbb{C}$ is open. To see this one has to show that $\widehat{V} := \pi^{-1}(\pi(V))$ is open. But

$$\widehat{V} = \bigcup_{\omega \in \Gamma} (\omega + V).$$

Since every set $\omega + V$ is open, so is \widehat{V} .

The complex structure on \mathbb{C}/Γ is defined in the following way. Let $V \subset \mathbb{C}$ be an open set such that no two points in V are equivalent under Γ . Then $U := \pi(V)$ is open and $\pi|_V : V \rightarrow U$ is a homeomorphism. Its inverse $\varphi : U \rightarrow V$ is a complex chart on \mathbb{C}/Γ . Let \mathcal{U} be the set of all charts obtained in this fashion. We have to show that any two charts $\varphi_i : U_i \rightarrow V_i$, $i = 1, 2$, belonging to \mathcal{U} are holomorphically compatible. Consider the map

$$\psi := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2).$$

For every $z \in \varphi_1(U_1 \cap U_2)$ one has $\pi(\psi(z)) = \varphi_1^{-1}(z) = \pi(z)$ and thus $\psi(z) - z \in \Gamma$. Since Γ is discrete and ψ is continuous, this implies that $\psi(z) - z$ is constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus ψ is holomorphic. Similarly ψ^{-1} is also holomorphic.

Now let \mathbb{C}/Γ have the complex structure defined by the complex atlas \mathcal{U} .

Remark 1.5. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. The map which associates to the point of \mathbb{C}/Γ represented by $\lambda\omega_1 + \mu\omega_2$, $(\lambda, \mu \in \mathbb{R})$, the point

$$(e^{2\pi i\lambda}, e^{2\pi i\mu}) \in S^1 \times S^1,$$

is a homeomorphism of \mathbb{C}/Γ onto the torus $S^1 \times S^1$.

Definition 1.6 (holomorphic function and structure sheaf on Riemann surface). Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f : Y \rightarrow \mathbb{C}$ is called **holomorphic**, if for every chart $\psi : U \rightarrow V$ on X the function

$$f \circ \psi^{-1} : \psi(U \cap Y) \longrightarrow \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$. The set of all functions holomorphic on Y will be denoted by $\mathcal{O}(Y)$.

Remark 1.6. 1. The sum and product of holomorphic functions are again holomorphic. Also constant functions are holomorphic. Thus $\mathcal{O}(Y)$ is a \mathbb{C} -algebra.

2. Of course the condition in the definition does not have to be verified for all charts in a maximal atlas on X , just for any family of charts covering Y . Then it is automatically fulfilled for all other charts.
3. Every chart $\psi : U \rightarrow V$ on X is, in particular, a complex-valued function on U . Trivially it is holomorphic. One also calls ψ a local coordinate or a uniformizing parameter and (U, ψ) a coordinate neighborhood of any point $a \in U$. In this context one generally uses the letter z instead of ψ .

Theorem 1.4 (Riemann's Removable Singularities Theorem). *Let U be an open subset of a Riemann surface and let $a \in U$. Suppose the function $f \in \mathcal{O}(U \setminus \{a\})$ is bounded in some neighborhood of a . Then f can be extended uniquely to a function $\tilde{f} \in \mathcal{O}(U)$.*

This follows directly from Riemann's Removable Singularities Theorem in the complex plane.

We now define holomorphic mappings between Riemann surfaces.

Definition 1.7 (holomorphic map). *Suppose X and Y are Riemann surfaces. A continuous mapping $f : X \rightarrow Y$ is called **holomorphic**, if for every pair of charts $\psi_1 : U_1 \rightarrow V_1$ on X and $\psi_2 : U_2 \rightarrow V_2$ on Y with $f(U_1) \subset U_2$, the mapping*

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \longrightarrow V_2$$

*is holomorphic in the usual sense. A mapping $f : X \rightarrow Y$ is called **biholomorphic** if it is bijective and both $f : X \rightarrow Y$ and $f^{-1} : Y \rightarrow X$ are holomorphic. Two Riemann surfaces X and Y are called **isomorphic** if there exists a biholomorphic mapping $f : X \rightarrow Y$.*

Remark 1.7. 1. In the special case $Y = \mathbb{C}$, holomorphic mappings $f : X \rightarrow \mathbb{C}$ are clearly the same as holomorphic functions.

2. If X , Y and Z are Riemann surfaces and $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are holomorphic mappings, then the composition $g \circ f : X \rightarrow Z$ is also holomorphic.
3. A continuous mapping $f : X \rightarrow Y$ between two Riemann surfaces is holomorphic precisely if for every open set $V \subset Y$ and every holomorphic function $\psi \in \mathcal{O}(V)$, the “**pull-back**” function $\psi \circ f : f^{-1}(V) \rightarrow \mathbb{C}$ is contained in $\mathcal{O}(f^{-1}(V))$.

In this way a holomorphic mapping $f : X \rightarrow Y$ induces a mapping

$$f^* : \mathcal{O}(V) \longrightarrow \mathcal{O}(f^{-1}(V)), \quad f^*(\psi) = \psi \circ f.$$

One can easily check that f^* is a ring homomorphism. If $g : Y \rightarrow Z$ is another holomorphic mapping, W is open in Z , $V := g^{-1}(W)$ and $U := f^{-1}(V)$, then

$(g \circ f)^* : \mathcal{O}(W) \rightarrow \mathcal{O}(U)$ is the composition of the mappings $g^* : \mathcal{O}(W) \rightarrow \mathcal{O}(V)$ and $f^* : \mathcal{O}(V) \rightarrow \mathcal{O}(U)$, i.e.,

$$(g \circ f)^* = f^* \circ g^*.$$

Theorem 1.5 (Identity Theorem). *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \rightarrow Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

Proof. Let G be the set of all points $x \in X$ having an open neighborhood W such that $f_1|_W = f_2|_W$. By definition G is open. We claim that G is also closed. For, suppose b is a boundary point of G . Then $f_1(b) = f_2(b)$ since f_1 and f_2 are continuous. Choose charts $\varphi : U \rightarrow V$ on X and $\psi : U' \rightarrow V'$ on Y with $b \in U$ and $f_i(U) \subset U'$. We may also assume that U is connected. The mappings

$$g_i := \psi \circ f_i \circ \varphi^{-1} : V \rightarrow V' \subset \mathbb{C}$$

are holomorphic. Since $U \cap G \neq \emptyset$, the Identity Theorem for holomorphic functions on domains in \mathbb{C} implies g_1 and g_2 are identically equal. Thus $f_1|_U = f_2|_U$. Hence $b \in G$ and thus G is closed. Now since X is **connected** either $G = \emptyset$ or $G = X$. But the first case is excluded since $a \in G$ (using the Identity Theorem in the plane again). Hence f_1 and f_2 coincide on all of X . \square

Definition 1.8 (meromorphic functions in Riemann surface). *Let X be a Riemann surface and Y be an open subset of X . By a **meromorphic function** on Y we mean a holomorphic function $f : Y' \rightarrow \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that the following hold:*

1. $Y \setminus Y'$ contains only isolated points.
2. For every point $p \in Y \setminus Y'$ one has

$$\lim_{x \rightarrow p} |f(x)| = \infty.$$

The points of $Y \setminus Y'$ are called the **poles** of f . The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

Remark 1.8. 1. Let (U, z) be a coordinate neighborhood of a pole p of f with $z(p) = 0$. Then f may be expanded in a Laurent series

$$f = \sum_{v=-k}^{\infty} c_v z^v$$

in a neighborhood of p .

2. $\mathcal{M}(Y)$ has the natural structure of a \mathbb{C} -algebra. First of all the sum and the product of two meromorphic functions $f, g \in \mathcal{M}(Y)$ are holomorphic functions at those points where both f and g are holomorphic. Then one holomorphically extends, using Riemann's Removable Singularities Theorem, $f+g$ (resp. fg) across any singularities which are removable.

Example 1.5. Suppose $n \geq 1$ and let

$$F(z) = z^n + c_1 z^{n-1} + \cdots + c_n, \quad c_k \in \mathbb{C},$$

be a polynomial. Then F defines a holomorphic mapping $F : \mathbb{C} \rightarrow \mathbb{C}$. If one thinks of \mathbb{C} as a subset of \mathbb{P}^1 , then $\lim_{z \rightarrow \infty} |F(z)| = \infty$. Thus $F \in \mathcal{M}(\mathbb{P}^1)$.

We now interpret meromorphic functions as holomorphic mappings into the Riemann sphere.

Theorem 1.6. Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f , define $f(p) := \infty$. Then $f : X \rightarrow \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f : X \rightarrow \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞ or else $f^{-1}(\infty)$ consists of isolated points and $f : X \setminus f^{-1}(\infty) \rightarrow \mathbb{C}$ is a meromorphic function on X .

From now on we will identify a meromorphic function $f \in \mathcal{M}(X)$ with the corresponding holomorphic mapping $f : X \rightarrow \mathbb{P}^1$.

Proof. 1. Let $f \in \mathcal{M}(X)$ and let P be the set of poles of f . Then f induces a mapping $f : X \rightarrow \mathbb{P}^1$ which is clearly continuous. Suppose $\varphi : U \rightarrow V$ and $\psi : U' \rightarrow V'$ are charts on X and \mathbb{P}^1 respectively with $f(U) \subset U'$. We have to show that

$$g := \psi \circ f \circ \varphi^{-1} : V \longrightarrow V'$$

is holomorphic. Since f is holomorphic on $X \setminus P$, it follows that g is holomorphic on $V \setminus \varphi(P)$. Hence by Riemann's Removable Singularities Theorem, g is holomorphic on all of V .

2. The converse follows from the Identity Theorem. □

Remark 1.9. The Identity Theorem also holds for meromorphic functions on a Riemann surface. Thus any function $f \in \mathcal{M}(X)$ which is not identically zero has only isolated zeros. This implies that $\mathcal{M}(X)$ is a field.

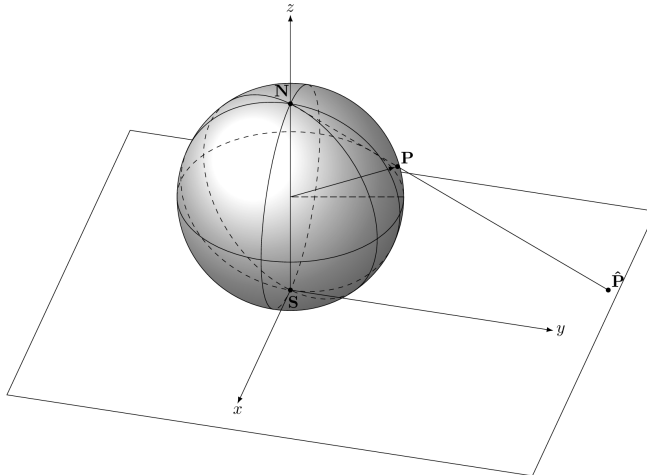


Figure 3: Riemann sphere

2 Mittag-Leffler problem, Weierstrass problem and the field of meromorphic functions on Riemann sphere \mathbb{P}^1

2.1 Riemann sphere \mathbb{P}^1 (or $\mathbb{C}P^1$)

Consider the Euclidean unit sphere in \mathbb{R}^3 , the set $\{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1\}$. Define the 2 charts:

$$V_1 = \{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1, Z > -\frac{3}{5}\}, \quad \phi_1 : (X, Y, Z) \mapsto \frac{X + iY}{1 + Z}.$$

$$V_2 = \{(X, Y, Z) \mid X^2 + Y^2 + Z^2 = 1, Z < \frac{3}{5}\}, \quad \phi_2 : (X, Y, Z) \mapsto \frac{X - iY}{1 - Z}.$$

Let U_1 (resp. U_2), the image of ϕ_1 (resp. ϕ_2), be the open disc $D(0, 2) \subset \mathbb{C}$. The image by ϕ_1 (resp. ϕ_2) of $V_1 \cap V_2$ is the annulus $\frac{1}{2} < |z| < 2$ in U_1 (resp. U_2). On this annulus, the transition map

$$\phi_2 \circ \phi_1^{-1} = \psi : z \mapsto \frac{1}{z}$$

is analytic, bijective, and its inverse is analytic. This defines the **Riemann sphere**, which is compact (the 2 charts are bounded discs $D(0, 2)$), connected and simply connected Riemann surface. The map ϕ_1 (resp. ϕ_2) is called the **stereographic projection** from the south (resp. north) pole of the sphere to the Euclidean plane $Z = 0$ in \mathbb{R}^3 , identified with \mathbb{C} .

Another definition of the Riemann sphere is the **complex projective plane** \mathbb{CP}^1 :

$$\mathbb{CP}^1 = \frac{\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid (z_1, z_2) \neq (0, 0)\}}{(z_1, z_2) \sim (\lambda z_1, \lambda z_2), \forall \lambda \in \mathbb{C}^*}.$$

It has also an atlas of 2 charts,

$$V_1 = \{[(z_1, z_2)] \mid z_2 \neq 0\}, \quad \phi_1 : [(z_1, z_2)] \mapsto z_1/z_2,$$

$$V_2 = \{[(z_1, z_2)] \mid z_1 \neq 0\}, \quad \phi_2 : [(z_1, z_2)] \mapsto z_2/z_1,$$

with transition map $z \mapsto 1/z$ (everything is well defined on the quotient by equivalent relation \sim). Thus \mathbb{CP}^1 is analytically isomorphic to the Riemann sphere previously defined.

Another definition of the Riemann sphere is from an abstract atlas of 2 charts

$$U_1 = D(0, R_1) \subset \mathbb{C} \quad \text{and} \quad U_2 = D(0, R_2) \subset \mathbb{C}$$

whose radii satisfy $R_1 R_2 > 1$. The 2 discs are glued by the analytic transition map

$$\psi : z \mapsto \frac{1}{z}$$

from the annulus $\frac{1}{R_2} < |z| < R_1$ in U_1 to the annulus $\frac{1}{R_1} < |z| < R_2$ in U_2 .

In other words, consider the following subset of $\mathbb{C} \times \{1, 2\}$

$$\frac{\{(z, i) \in \mathbb{C} \times \{1, 2\} \mid z \in U_i\}}{\sim}.$$

quotiented by the equivalence relation

$$(z, i) \sim (\tilde{z}, j) \iff (i = j \text{ and } z = \tilde{z}) \text{ or } (i + j = 3 \text{ and } z\tilde{z} = 1).$$

This Riemann surface is analytically isomorphic to the Riemann sphere previously defined.

Notice that one can choose R_1 very large, and R_2 very small, and even consider a projective limit $R_1 \rightarrow \infty$ and $R_2 \rightarrow 0$, in other words glue the whole $U_1 = \mathbb{C}$ to the single point $U_2 = \{0\}$. Notice that the point $z' = 0$ in U_2 should correspond to the point $z = 1/z' = \infty$ in $U_1 = \mathbb{C}$. In this projective limit, by adding a single point to \mathbb{C} , we turn it into a compact Riemann surface $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The topology of $\overline{\mathbb{C}}$ is generated by the open sets of \mathbb{C} , as well as all the sets

$$V_R = \{\infty\} \cup \{z \in \mathbb{C} \mid |z| > R\} \quad \text{for all } R \geq 0.$$

These open sets form a basis of neighborhoods of ∞ . With this topology, $\overline{\mathbb{C}}$ is compact.

This justifies that the Riemann sphere is called a compactification of \mathbb{C} :

$$\mathbb{CP}^1 = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

2.2 $\text{Aut}(\mathbb{P}^1)$ and fractional linear transformations on \mathbb{P}^1

Definition 2.1 (automorphism group: $\text{Aut}(X)$). *Let X be a Riemann surface,*

$$\text{Aut}(X) = \{\varphi : X \rightarrow X \mid \varphi \text{ is biholomorphic}\},$$

where “biholomorphic” means holomorphic, bijective, and with holomorphic inverse.

Remark 2.1. *Conformal equivalence and biholomorphism for any complex dimensions are the same.*

Definition 2.2 (Conformal equivalence). *Two Riemann surfaces X, Y are **conformally equivalent** (or **biholomorphically equivalent**) if there is a biholomorphism $\varphi : X \rightarrow Y$.*

Definition 2.3 (Fractional linear (Möbius) transformation). *Let $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, we call*

$$\varphi(z) = \frac{az + b}{cz + d},$$

a fractional linear (Möbius) transformation.

Remark 2.2. *If $c = d = 0$ then $ad - bc = 0$, so that case is not allowed.*

We extend φ from \mathbb{C} to the sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the following natural conventions:

- (Regular points) If $z_0 \in \mathbb{C}$ and $cz_0 + d \neq 0$, set

$$\varphi(z_0) = \frac{az_0 + b}{cz_0 + d} \in \mathbb{C}.$$

- (Poles) If $z_0 \in \mathbb{C}$ and $cz_0 + d = 0$ (equivalently $z_0 = -\frac{d}{c}$ with $c \neq 0$), then $az_0 + b = -\frac{ad}{c} + b = \frac{-ad+bc}{c}$ is constant, set

$$\lim_{z \rightarrow z_0} \varphi(z) = \lim_{z \rightarrow z_0} \left| \frac{az + b}{cz + d} \right| = \infty,$$

so we define $\varphi(z_0) = \infty \in \mathbb{P}^1$

- (At infinity) For $z_0 = \infty$, we have

$$\varphi(\infty) = \lim_{z \rightarrow \infty} \varphi(z) = \lim_{z \rightarrow \infty} \frac{az + b}{cz + d} = \lim_{z \rightarrow \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \rightarrow \begin{cases} \frac{a}{c}, & c \neq 0, \\ \infty, & c = 0. \end{cases}$$

The set of all the fractional linear transformations on \mathbb{P}^1 forms a group, with the multiplication operation given by composition.

Proposition 2.1 ($\text{FractLin}(\mathbb{P}^1)$). *The group of all the fractional linear transformations on \mathbb{P}^1 :*

$$\text{FractLin}(\mathbb{P}^1) := \left\{ \frac{az + b}{cz + d} : ad - bc \neq 0 \right\}.$$

Proof. • (Closure) For $\varphi(z) = \frac{az + b}{cz + d}$ and $\psi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$,

$$(\varphi \circ \psi)(z) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)},$$

which is again fractional linear. Its determinant equals $(ad - bc)(\alpha\delta - \beta\gamma) \neq 0$.

- (Associativity) Follows from associativity of function composition, equivalently, matrix multiplication is associative:

$$(\varphi \circ \psi) \circ \theta = \varphi \circ (\psi \circ \theta) \iff (AB)C = A(BC).$$

- (Identity)

$$\text{id}(z) = z = \frac{1 \cdot z + 0}{0 \cdot z + 1} \iff I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

lies in the set.

- (Inverse) If $\varphi(z) = \frac{az + b}{cz + d}$, then

$$\varphi^{-1}(z) = \frac{dz - b}{-cz + a},$$

corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, hence $\varphi^{-1} \in \text{FractLin}(\mathbb{P}^1)$. □

The correspondence between the composition of linear transformations and matrix multiplication: take

$$\varphi(w) = \frac{aw + b}{cw + d}, \quad \psi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad ad - bc \neq 0, \quad \alpha\delta - \beta\gamma \neq 0.$$

Compute

$$(\varphi \circ \psi)(z) = \frac{a \frac{\alpha z + \beta}{\gamma z + \delta} + b}{c \frac{\alpha z + \beta}{\gamma z + \delta} + d} = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}.$$

Hence the composition is again fractional linear: closure holds.

Associate

$$\varphi \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \psi \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and

$$(\varphi \circ \psi)(z) = \frac{Az + B}{Cz + D}.$$

Determinants multiply:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so the composite is nondegenerate, because assumption $ad \neq cd$ means $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Finally, under composition, $\text{FractLin}(\mathbb{P}^1)$ is a group. Via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az + b}{cz + d} \right),$$

this group is isomorphic to

$$\text{PGL}(2, \mathbb{C}) = \text{GL}(2, \mathbb{C}) / \mathbb{C}^*$$

Lemma 2.1 (Riemann removable singularity). *If f is holomorphic on a punctured neighborhood $0 < |z - z_0| < r$ and bounded near z_0 , then f extends holomorphically to z_0 .*

Lemma 2.2 (Generalized Liouville). *If an entire function f satisfies $|f(z)| \leq A|z|^m + B$ for large $|z|$, then f is a polynomial of degree $\leq m$.*

Theorem 2.1. *Automorphism group on the Riemann Sphere: $\text{Aut}(\mathbb{P}^1) = \text{FractLin}(\mathbb{P}^1)$, i.e., every automorphism of the Riemann sphere $\mathbb{P}^1 = \widehat{\mathbb{C}}$ is a Möbius (fractional linear) transformation.*

Proof. Hint:

The inclusion “ \supset ” is clear: Möbius maps are biholomorphic on \mathbb{P}^1 . The main task is “ \subset ”: for any $\varphi \in \text{Aut}(\mathbb{P}^1)$, want to show φ is a fractional linear transformation.

1. If $\varphi(\infty) = \infty$, then $\varphi|_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ is an automorphism of \mathbb{C} , i.e. $\varphi|_{\mathbb{C}} \in \text{Aut}(\mathbb{C})$,
2. If $\varphi(\infty) = s \in \mathbb{C}$, compose with $\psi(w) = \frac{1}{w - s}$ so that $\psi(s) = \infty$ and $\psi \in \text{FractLin}(\mathbb{P}^1)$, then set $\theta = \psi \circ \varphi \in \text{FractLin}(\mathbb{P}^1)$, $\theta(\infty) = (\psi \circ \varphi)(\infty) = \infty$,

reducing to case 1: $\theta|_{\mathbb{C}} \in \text{Aut}(\mathbb{C})$, finally $\varphi = \psi^{-1} \circ \theta \in \text{FractLin}(\mathbb{P}^1)$.

so proving this theorem suffices to show $\text{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a \neq 0\}$.

Let $\varphi \in \text{Aut}(\mathbb{C})$ and φ is continuous at ∞ , consider another local coordinate on a punctured neighborhood $0 < |z - \infty| < r$ with the transition map $w = \frac{1}{z}$ and

$$\tilde{\varphi}(w) := \frac{1}{\varphi(1/w)}.$$

Take $z' = \varphi(z)$ and $w' = \tilde{\varphi}(w)$, so we have $zw = 1$ in \mathbb{P}^1 and $z'w' = 1$ in $\varphi(\mathbb{P}^1)$, where φ and $\tilde{\varphi}$ are same maps with different coordinates.

φ is continuous at ∞ , so when $|z| \rightarrow \infty$, $\varphi(z)$ is bounded, that implies $\tilde{\varphi}$ is bounded near $w = 0$, then by the **Removable Singularity Theorem 2.1** it extends holomorphically at $w = 0$; equivalently, φ behaves well at ∞ and extends to a holomorphic map on \mathbb{P}^1 :

$$\lim_{w \rightarrow 0} \tilde{\varphi}(w) = \lim_{w \rightarrow 0} \frac{1}{\varphi(1/w)} = \lim_{z \rightarrow \infty} \frac{1}{\varphi(z)} = \frac{1}{\varphi(\infty)} = \frac{1}{\infty} = 0 \implies \tilde{\varphi}(0) = 0.$$

Take the Taylor expansion

$$\tilde{\varphi}(w) = c_1 w + c_2 w^2 + \cdots + c_n w^n + \cdots, \quad c_1 \neq 0.$$

Thus

$$\begin{aligned} \varphi(z) = z' = \frac{1}{w'} &= \frac{1}{\tilde{\varphi}(w)} = \frac{1}{c_1 w + c_2 w^2 + \cdots}, \quad c_1 \neq 0 \\ &= \frac{1}{c_1} \left(\frac{1}{w + \frac{c_2}{c_1} w^2 + \frac{c_3}{c_1} w^3 + \cdots} \right), \quad c_1 \neq 0 \\ &= \frac{1}{c_1 w} \left(\frac{1}{1 + \frac{c_2}{c_1} w + \frac{c_3}{c_1} w^2 + \cdots} \right), \quad c_1 \neq 0 \end{aligned}$$

implies that $\exists C$ such that $\varphi(z) = z'$ satisfies

$$\varphi(z) = \frac{1}{\tilde{\varphi}(1/z)} = \frac{1}{c_1(1/z) + c_2(1/z)^2 + \cdots} = \frac{z}{c_1} \left(1 + \mathcal{O}(1/z) \right),$$

so there are constants $C, C' > 0$ with

$$|\varphi(z)| \leq C|z| + C' \quad (|z| \text{ large}).$$

Finally, by the **generalized Liouville theorem** (via Cauchy estimates) 2.2, let $m = 1$, we have: $\exists a, b \in \mathbb{C}$ such that for $\forall z \in \mathbb{C} : \varphi(z) = az + b$. φ is injective if and only if $a \neq 0$. Hence

$$\text{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \neq 0\}.$$

□

2.3 Mittag-Leffler problem on \mathbb{P}^1

Definition 2.4 (Principal part at a point). *Let X be a Riemann surface and $a \in X$. Choose a local coordinate t with $t(a) = 0$. If f is meromorphic near a ,*

$$f(t) = \sum_{k=-m}^{\infty} c_k t^k,$$

*the **principal part** of f at a is the finite sum of the negative powers:*

$$\text{PP}_a(f) := \sum_{k=1}^m c_{-k} t^{-k}.$$

It is independent of the chosen coordinate.

Remark 2.3. *When $a = \infty$ on \mathbb{P}^1 , take $w = 1/z$; the principal part at ∞ is the negative-power part of the Laurent expansion in w , equivalently a polynomial in z :*

$$\text{principal part at } \infty = \sum_{i=1}^s c_i w^{-i} = \sum_{i=1}^s c_i z^i \quad (a \text{ polynomial in } z).$$

Theorem 2.2 (Mittag-Leffler problem on Riemann surface \mathbb{P}^1). *Given finitely many points $a_1, \dots, a_n \in \mathbb{P}^1$ and principal-part data p_1, \dots, p_n at those points, find a meromorphic function f such that*

$$\text{PP}_{a_k}(f) = p_k, \quad \forall k$$

and f is holomorphic away from these points. One may also assume the a_k are pairwise distinct (combine the data at coincident points), where pairs $\{(a_1, p_1), \dots, (a_n, p_n)\}$ called Mittag-Leffler data.

Theorem 2.3. *Mittag-Leffler problem is always solvable on \mathbb{P}^1 .*

Proof. For any finite set of points a_1, a_2, \dots, a_n (allowing ∞) and any principal parts p_1, p_2, \dots, p_n at a_1, a_2, \dots, a_n , respectively. We may take a_1, a_2, \dots, a_n to be distinct.

Write the principal part at each point a_k as

$$p_k(z) = \sum_{i=1}^{s_k} \frac{c_{k,i}}{(z - a_k)^i}, \quad s_k \in \mathbb{Z}_{>0}.$$

1. Case: $\infty \notin \{a_1, \dots, a_n\}$

We have: for $k \neq l$, p_k has a pole only at a_k and is holomorphic at a_l , in fact p_k is holomorphic on $\mathbb{C} - \{a_k\}$.

Define

$$f(z) = \sum_{k=1}^n p_k(z) = \sum_{k=1}^n \sum_{i=1}^{s_k} \frac{c_{k,i}}{(z - a_k)^i}.$$

easy to check that f is holomorphic on $\mathbb{C} \setminus \{a_1, \dots, a_n\}$. At a_k , the principal part of f is precisely p_k .

Claim: f solve the M-L problem for the M-L data $\{(a_1, p_1), \dots, (a_n, p_n)\}$.

To prove the claim, it remains to check that f is holomorphic at ∞ .

Check: $\forall k$,

$$\lim_{z \rightarrow \infty} p_k(z) = \lim_{z \rightarrow \infty} \sum_{i=1}^{s_k} \frac{c_{k,i}}{(z - a_k)^i} = \mathcal{O}(|z|^{-1}) \Rightarrow p_k(z) \rightarrow 0$$

hence

$$\lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} \sum_{k=1}^n p_k(z) = 0$$

so f extends holomorphically at ∞ (with value 0). Therefore f is meromorphic on \mathbb{P}^1 and solves the ML problem

2. Case: $\infty \in \{a_1, \dots, a_n\}$ and $\{a_1, \dots, a_n\}$ pairwise distinct, denoted by E .

Without loss of generality set $a_n = \infty$. Use coordinate $w = 1/z$ to write the principal part at ∞ :

$$\tilde{p}_n(w) = \sum_{i=1}^{s_n} \frac{c_{n,i}}{w^i} = \sum_{i=1}^{s_n} c_{n,i} z^i =: p_n(z) \quad (\text{a polynomial}).$$

$w = 0$ at point $\infty \in \mathbb{P}^1$

Now define

$$f(z) = \sum_{k=1}^{n-1} p_k(z) + p_n(z), \text{ where } p_n(z) = \tilde{p}_n\left(\frac{1}{z}\right).$$

For $k \leq n-1$, $p_k \rightarrow 0$ at ∞ , so the principal part of f at ∞ is entirely determined by $p_n(z)$, exactly matching the given \tilde{p}_n . At each finite a_k , the principal part is still p_k . Thus f solves the ML data including ∞ .

□

2.4 Weierstrass problem on \mathbb{P}^1

Let $E = \{a_1, \dots, a_s\} \subset \mathbb{P}^1$ be a finite set of distinct points. For each k ($1 \leq k \leq s$) give an integer n_k . The collection $\{(a_1, n_1), \dots, (a_s, n_s)\}$ is called a **Weierstrass data set**.

Theorem 2.4 (Weierstrass (divisor prescription) problem on \mathbb{P}^1). *Given a finite set $E = \{a_1, \dots, a_s\} \subset \mathbb{P}^1$ and integers n_1, \dots, n_s . Question: does there exist a meromorphic function f on \mathbb{P}^1 , such that*

$$\text{ord}_{a_k}(f) = n_k \quad (1 \leq k \leq s),$$

and f has no other zeros/poles? (If one $a_k = \infty$, interpret ord_∞ in the local coordinate $w = 1/z$.)

Definition 2.5 (Divisor and degree). *For a meromorphic f on \mathbb{P}^1 , its divisor*

$$\text{div}(f) = \sum_{p \in \mathbb{P}^1} \text{ord}_p(f) \cdot p$$

has degree

$$\deg \text{div}(f) = \sum_p \text{ord}_p(f).$$

Definition 2.6 (order). $\text{ord}_x(f)$

- *If x is a zero of order $m > 0$, then $\text{ord}_x(f) = m$;*
- *if x is a pole of order $m > 0$, then $\text{ord}_x(f) = -m$; otherwise $\text{ord}_x(f) = 0$.*

At ∞ we use the local coordinate $w = 1/z$: $\text{ord}_\infty(f) := \text{ord}_{w=0}(f(1/w))$.

Theorem 2.5. *For the given Weierstrass data set $\{(a_1, n_1), \dots, (a_s, n_s)\}$, Weierstrass problem is solvable if and only if $n_1 + \dots + n_s = 0$.*

Remark 2.4. *If we switch to the coordinate $w = 1/z$, the circle parametrization changes from $z = Re^{i\theta}$ to $w = \frac{1}{R}e^{-i\theta}$ (orientation reversal)*

Proof. • **Necessity:**

Argument principle: Let $f \in \mathcal{M}(\mathbb{P}^1)$, $f \not\equiv 0$. Without loss of generality, we may assume the a_k 's do not lie on the unit circle, or we can in any case choose R so that $\partial D(R) \cap E = \emptyset$, where $D(R) = D(0; R)$.

Let $f(z) = (z - a)^n h(z)$, where $h(z)$ is a holomorphic function at point $z = a$, then $\log f = n \log(z - a) + \log h$, calculate the derivative:

$$\begin{aligned} \frac{f'}{f} &= (\log f)' = n (\log(z - a))' + (\log h)' \\ &= \frac{n}{z - a} + \frac{h'}{h} \end{aligned}$$

hence

$$\text{Res} \left(\frac{f'}{f}, a \right) = n = \text{ord}_a(f)$$

Counting function of zeros (with multiplicity) of $f \in \mathcal{M}(\mathbb{P}^1)$, $f \not\equiv 0$:

$$\begin{aligned} \# \text{Zeros}(f|_{D(R)}) - \text{Res}(g, a) &= \text{ord}_a(g), \quad \text{where } g = \frac{f'}{f} = (\log f)' \\ &= \frac{1}{2\pi i} \oint_{\partial D(R)} \frac{f'(\zeta)}{f(\zeta)} d\zeta, \quad \text{by argument principle} \end{aligned}$$

In another words,

$$\#\{\text{zeros of } f \text{ in } D(R)\} - \#\{\text{poles of } f \text{ in } D(R)\} = \frac{1}{2\pi i} \oint_{\partial D(R)} \frac{f'(\zeta)}{f(\zeta)} d\zeta,$$

and $\text{Res}(g, a) = \text{ord}_a(f)$.

Include the exterior region as well (inversion coordinate and orientation change).

In the coordinate $\xi = 1/\zeta$, the circle $|\zeta| = R$ maps to $|\xi| = 1/R$ with reversed orientation (as on your board: $z = Re^{i\theta}$ corresponds to $w = \frac{1}{R}e^{-i\theta}$). Set

$$S(\xi) := f\left(\frac{1}{\xi}\right).$$

Then

$$\oint_{\partial D(R)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \oint_{\partial D(R)} d\log f(\zeta) = - \oint_{\partial D(\frac{1}{R})} d\log S(\xi) = - \oint_{\partial D(\frac{1}{R})} \frac{S'(\xi)}{S(\xi)} d\xi.$$

Apply the residue theorem on both sides:

$$\begin{aligned} LHS &= \oint_{\partial D(R)} d\log f = 2\pi i \sum_{a_i \in D(R)} \text{Res}\left(\frac{f'}{f}, a_i\right) = 2\pi i \sum_{a_i \in D(R)} \text{ord}_{a_i} f, \\ RHS &= - \oint_{\partial D(\frac{1}{R})} d\log S = -2\pi i \sum_{a_j \in \mathbb{P}^1 \setminus \overline{D(R)}} \text{Res}(d\log f, a_j) \\ &= -2\pi i \sum_{a_j \in D(\frac{1}{R})} \text{Res}\left(\frac{S'}{S}, a_j\right) = -2\pi i \sum_{a_j \notin D(R)} \text{ord}_{a_j} f. \end{aligned}$$

Equating and combining the two gives

$$0 = \sum_{a_i \in D(R)} \text{Res}(d\log f, a_i) + \sum_{a_j \in \mathbb{P}^1 \setminus \overline{D(R)}} \text{Res}(d\log f, a_j) = \sum_{k=1}^s \text{ord}_{a_k} f.$$

where

$$\text{ord}_a f = \begin{cases} \text{order of zero at } a \\ - \text{order of pole at } a \\ 0 \text{ when holomorphic and nonzero at } a \end{cases}$$

That is, Weierstrass problem solvable for data $\{(a_1, n_1), \dots, (a_s, n_s)\}$ implies

$$\sum_{k=1}^s n_k = 0$$

• **Sufficiency** (by construction)

1. Case: $\infty \notin E$

Define

$$f(z) = \prod_{k=1}^s (z - a_k)^{n_k}$$

(negative exponents understood in the denominator). Then

$$\forall k \in \{1, 2, \dots, s\} : \text{ord}_{a_k} f = n_k$$

$$\text{ord}_a f = 0 \quad (a \in \mathbb{C} \setminus E).$$

hence f has the prescribed order of zeros/poles at any finite point $x \in \mathbb{C}$

At ∞ we have

$$\text{ord}_\infty f = - \sum_{k=1}^s n_k.$$

If the solvability condition $\sum n_k = 0$ holds, then $\text{ord}_\infty f = 0$. Hence f has the prescribed orders at all finite points and no zero/pole at ∞ : construction succeeds.

Write

$$f(z) = \prod_{k=1}^{s-1} (z - a_k)^{n_k}$$

where $\sum_{k=1}^{s-1} n_k = -n_s$, we have $\forall a \in \mathbb{C}$:

$$\begin{cases} \text{ord}_{a_k} f = n_k \\ \text{ord}_a f = 0, \quad \forall a \in \mathbb{C} - \{a_1, a_2, \dots, a_s\} \end{cases}$$

2. Case: $\infty \in E$, say $a_s = \infty$

Now the condition reads

$$\sum_{k=1}^{s-1} n_k = -n_s.$$

Define

$$f(z) = \prod_{k=1}^{s-1} (z - a_k)^{n_k}$$

For each finite point a_k ($k \leq s-1$) we have $\text{ord}_{a_k} f = n_k$. At ∞ ,

$$f(z) = \prod_{k=1}^{s-1} z^{n_k} \left(1 - \frac{a_k}{z}\right)^{n_k} = z^{\sum_{k=1}^{s-1} n_k} \prod_{k=1}^{s-1} \left(1 - \frac{a_k}{z}\right)^{n_k} = z^{-n_s} h(z),$$

where $\lim_{z \rightarrow \infty} h(z) = 1$. Hence

$$\text{ord}_{\infty} f = \text{ord}_{\infty} (z^{-n_s}) + \text{ord}_{\infty} h = \text{ord}_{w=0} (w^{n_s}) + \text{ord}_{\infty} h = n_s + 0 = n_s.$$

Therefore the order at $\infty = a_s$ is also as prescribed, and the construction is complete. □

2.5 $\mathcal{M}(\mathbb{P}^1)$: the field of all the meromorphic functions on \mathbb{P}^1

Definition 2.7 (Rational functions on \mathbb{P}^1). $f(z) = \frac{P(z)}{Q(z)}$, $P, Q \in \mathbb{C}[z]$, $Q \not\equiv 0$, allowed to have poles at finitely many points (including $z = \infty$). The set of all rational functions on \mathbb{P}^1 is denoted by $\text{Rat}(\mathbb{P}^1)$

Lemma 2.3 (Maximum Modulus Principle on Compact Riemann surface). *If X is a compact Riemann surface (e.g. \mathbb{P}^1) and $g \in \mathcal{O}(X)$, then g is constant. More generally, if $g \in \mathcal{M}(X)$ has ***no zeros and no poles***, then g is constant (since both g and $1/g$ are holomorphic).*

Theorem 2.6 ($\mathcal{M}(\mathbb{P}^1)$: the field of all the meromorphic functions on \mathbb{P}^1). $\mathcal{M}(\mathbb{P}^1) = \text{Rat}(\mathbb{P}^1)$. *That is, every meromorphic function on the Riemann sphere \mathbb{P}^1 is a rational function, and vice versa.*

Proof. As a corollary of the solution of Weierstrass problem.

For any nonzero meromorphic function f on \mathbb{P}^1 , $\sum'_{x \in \mathbb{P}^1} \text{ord}_x(f) = 0$, where the prime \sum' indicates the sum is over the finitely many relevant points (zeros and poles).

This is the argument principle on the sphere: the degree of the divisor of a nonzero meromorphic function is zero (total zero order equals total pole order, counting ∞).

Let the **Weierstrass data** of f be

$$\{(a_1, n_1), \dots, (a_s, n_s)\}, \quad a_k \in E = \text{Zeros}(f) \sqcup \text{Poles}(f), \quad n_k = \text{ord}_{a_k} f \in \mathbb{Z}.$$

By the argument principle, $\sum_{k=1}^s n_k = 0$.

By the solution of the Weierstrass problem on \mathbb{CP}^1 , there exists

$$h \in \text{Rat}(\mathbb{P}^1)$$

such that $\text{ord}_{a_k} h = n_k$ for all k and $\text{ord}_x h = 0$ for $x \notin E$ (solve the Weierstrass problem with Weierstrass data $\{(a_1, n_1), \dots, (a_s, n_s)\}$).

Then

$$\frac{f}{h} \in \mathcal{M}(\mathbb{P}^1) \quad \text{and} \quad \forall x \in \mathbb{P}^1 : \text{ord}_x \left(\frac{f}{h} \right) = \text{ord}_x f - \text{ord}_x h = 0.$$

Hence f/h is a meromorphic function without zeros or poles; by the **Maximum Principle 2.3** on the compact surface \mathbb{CP}^1 , f/h is a nonzero constant c . Therefore

$$f = ch \in \text{Rat}(\mathbb{CP}^1),$$

which proves $\mathcal{M}(\mathbb{CP}^1) = \text{Rat}(\mathbb{CP}^1)$. □